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Abstract: In this report, we construct analytical solutions and study numerically the well-posedness of several problems concerning the isotropic poroelastic equation in axisymmetry. The problems in consideration are cylindrical bounded domains, the scattering of plane wave in poroelastic by penetrable/impenetrable circular obstacles and lastly fluid-solid interaction problem with circular solid obstacles for closed, open and intermediate pore boundary type. Since we have analytic expressions for the coefficients / transmission matrices, well-posedness is investigated by probing for zeros of the determinant of these matrices. Our investigation includes the effect of material parameters and different degrees of viscosity. The first novelty of the work is the proposal of a definition of outgoing solutions for isotropic poroelasticity. The second novelty is the observation that there are modes in fluid-poroelastic interaction problems without viscosity which are the equivalent of Jones' modes for fluid-elastic problem, and that these modes cease to exist in the presence of viscosity. We found out that the presence of viscosity removes the eigenvalues, which exist without viscosity and whose existence is expected for bounded domains.

Key-words: outgoing solution, isotropic poroelasticity, analytic solution, fluid-solid interaction, attenuation, Jones modes, scattering of plane wave, eigenvalues.

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Valeurs propres de solutions analytiques et de problèmes de transmission dans des milieux poroélastiques isotropes pour des domaines bornés, pour la diffraction d'obstacles et pour des intéractions fluide-solide en 2D.

Résumé : Dans ce rapport, nous étudions les équations poroélastiques isotropes axisymétriques dans différentes configurations pour lesquelles nous avons construit des solutions analytiques et étudié numériquement que le problème est bien posé. Nous nous sommes intéréssés à des domaines cylindriques bornés, à la diffraction d'une onde plane dans un milieu poro-élastique par des obstacles circulaires pénétrables et impénétrables, et enfin à l'interaction fluide-solide. Puisque nous avons obtenu les expressions analytiques des matrices de coefficients, nous pouvons étudier le déterminant de ces matrices pour déterminer le caractère bien-posé du problème. Notre étude inclut l'effet des paramètres du matériau et de la valeur de la viscosité. La première nouveauté de ce travail est la proposition d'une définition de solutions sortantes pour des problèmes poro-élastiques isotropes. Deuxièmement, nous avons observé des modes pour des problèmes d'interaction fluides-poroélastiques, équivalents aux modes de Jones, qui ne sont pas présents quand le milieu est visqueux. De même, nous avons noté que la présence de viscosité élimine les valeurs propres, qui existent pourtant sans viscosité et sont attendues pour des domaines bornés.

Mots-clés : solution sortante, poroélasticité isotrope, solution analytique, intéraction fluide-solide, atténuation, mode de Jones, diffraction d'onde plane, valeurs propres.

1 Introduction

In this report, we consider the isotropic poroelastic equations in 2D. These are obtained from 3D problem with axis-symmetry. The equations are given by the linear theory of deformation of a porous medium, called the theory of consolidation, first created by Biot for the isotropic case in [4] [5]. This project initially grows out of the need for analytical solutions in order to evaluate the accuracy of the discretization of poroelastic equations by Hybridizable Discontinuous Galerkin method. However, the scope extends to cover not only homogeneous poroelastic equation on bounded domains, but also the scattering of plane wave by impenetrable and penetrable infinite cylindrical (thus with circular 2D cross-section) obstacles. The last group is also called fluid-solid interaction and deals with infinite cylindrical obstacles. This is opposed to horizontally stratified fluid-solid or solid-solid interaction problem, *cf. e.g.* [37, 25, 12, 13, 14], or to problems dealing with impulse / point sources. To obtain analytic solutions, we employ the potential method used by [27] in elasticity, which exploits the very specific form of the poroelastic equation and provides a lighter exposition than the usual approach with Helmholtz decomposition. In addition to the computation of analytic solution for each considered problem, we go further and propose a definition of outgoing solutions cf. Definition 2, and investigate numerically the well-posedness for interaction problems among others.

Current works in literature dealing with analytic solutions to the poroelastic equation construct fundamental solutions either for infinite domain, *cf. e.g.* [6], [7] (and thus deal with point sources), or horizontally stratified domain [13, 14]. For these reasons, they do not directly provide analytic solutions for plane wave scattering in spherical and cylindrical geometries, as were done for the elastic equation, *cf. e.g.* [29]. While the form of generic solutions to the homogeneous equation for infinite domain can be extracted from calculations of the fundamental solution in [6], [7] or in [13, 14], this approach can quickly become complicated, due to the multitude of poroelastic physical parameters whose notations and conventions vary with each work. Additionally, for scattering problem with plane waves, with zero right-hand-side terms, the form of the solution should be much simpler, and an adapted computation for this problem is not quite in the same vein as one employed to compute the fundamental solutions. Lastly, we work with dynamic viscocity [31] which depends on frequency, while [13, 14] work with a low-frequency approximation of the isotropic poroelastic equation and with vanishing viscosity. Our geophysical parameters are based on those in [18, 19, 11, 20].

The topics of outgoing solution and transmission eigenvalues for poroelasticity are not yet covered in literature. While the notion of outgoing solution is well-established for elasticity with the Kudrapze radiation condition [24], there does not seem to exist one for poroelasticity. Similarly, while the well-posedness of the interaction problem for acoustic fluid-elastic solid and the phenomenon of Jones' modes are covered in *e.g.* [16, 2, 23], this is not yet investigated for isotropic poroelasticity. From our numerical investigations, in this report, we detect the equivalent of Jones' modes for fluid-poroelasticity interaction problems with cylindrical obstacles in the absence of viscosity, however they cease to exist when there are viscosity. As mentioned above, we work with a frequency-dependent viscosity, and carry out various tests to study the effect of frequency, material parameters and viscosity on the well-posedness of the problem. This study paves the way for theoretical future investigations of questions, such as the well-posedness of the outgoing solutions and theoretical confirmation of Jones' modes for fluid-poroelasticity.

The organization of the report is as follows. We describe in Section 2 the physical parameters of the isotropic poroelastic equations and their meaning. In Section 3 we introduce the poroelastic equations in time and in frequency domain. Section 4 gives the explicit expression of a planewave sustained in an isotropic poroelastic medium. This is also important in computing the wave speeds occurring in such a medium. In Section 6, we use potential theory to reduce the poroelastic system to a set of Helmholtz equations, and the original poroelastic unknowns are now expressed in terms of the potentials which solve Helmholtz equations. Since these are constant coefficients Helmholtz equations, the potential, and thus the poroelastic unknowns, can be expressed in terms of Bessel functions. We apply these results to obtain analytical solutions for the following four settings: bounded domain in Section 7, impenetrable obstacles in Section 8, penetrable obstacles in Section 9, and fluid-solid interaction in Section 10. For each case, we first present detailed expressions of the solutions, which are obtained by solving a linear system, and we then numerically study the invertibility of the coefficient/ transmission matrices (of the aforementioned linear system). Finally, an overall comparison among the interaction problems is given in Section 11.

2 Physical Parameters

A porous medium is composed by a solid frame, and pores filled with a fluid. The Biot's model can be used when the following hypotheses are satisfied :

- The size of the pores is small in comparison with the wavelength.
- The displacements in the solid and fluid phases are small.
- The fluid phase is continuous.
- The solid frame is elastic.
- The thermo-mechanical effects are neglected.

We define the *porosity* as the ratio of the fluid volume and the total volume

$$\phi = \frac{V_f}{V_T} \,. \tag{2.1}$$

The geometry of pores is described in terms of *tortuosity* α_∞. The fluid is defined by an *uncompressibility modulus* $\mathrm{k_f}$, a *fluid density* ρ_f, the *viscosity* η and the *permeability* κ_0. The solid frame is defined by an *uncompressibility modulus* $\mathrm{k_s}$, the *solid density* ρ_s, an *uncompressibility drained modulus* $\mathrm{k_{fr}}$, a *shear modulus* μ_fr and a *consolidation parameter.* The *average density* of a poroelastic medium is defined as

$$\rho_a \;:=\; (1-\phi)\,\rho_\mathrm{s} \;+\; \phi\,\rho_\mathrm{f} \,. \tag{2.2}$$

The solid skeleton has compressibility and shearing rigidity, and the fluid can be compressible. To describe a porous medium, we use an homogenisation on the fluid and solid phases, to obtain an equivalent medium. For $\bullet = \mathrm{f},\ \mathrm{fr}$ and s, corresponding respectively to the fluid, the frame, and the solid, the relation between the bulk modulus $\mathsf{k}_\bullet$ and the Lamé parameters $\lambda_\bullet$, $\mu_\bullet$ is

$$\lambda_\bullet = \mathrm{k}_\bullet - \tfrac{2}{3}\mu_\bullet \quad .$$

We can consider two different conditions for the medium, drained or undrained. For the current discussion, we follow [9] and [10]. In undrained conditions, the solid is wrapped in a membrane and the fluid cannot flow out, or the fluid is viscous, and with a small amount of time, the fluid, does not flow out. In this case, there is a difference of pressure during the experiment, but no relative variation of fluid content ($\zeta = 0$). The moduli associated to this state are called the undrained ones, denoted by $\mu_\mathrm{undrained}$, $\lambda_\mathrm{undrained}$ and $\mathrm{k_{undrained}}$. They are also called Gassmann modulus,

$$\mu_G = \mu_\mathrm{undrained}\,, \quad \lambda_G = \lambda_\mathrm{undrained}\,, \quad \text{and } \mathrm{k}_G = \mathrm{k_{undrained}}\,.$$

A material has a drained response when the solid surface is exposed to the atmosphere, the fluid in the pores can flow out, but there is no variation of pressure inside the pores ($\Delta\mathrm{p} = 0$). The moduli associated to this state are denoted by μ_fr, λ_fr and $\mathrm{k_{fr}}$, also called the bulk modulus of the dry matrix or dry frame. The relations between the drained and undrained states are given by

$$\mu_G = \mu_\mathrm{fr}\,, \quad \text{and } \lambda_G = \lambda_\mathrm{fr} + \alpha^2 M \,. \tag{2.3}$$

In the above expression, the *effective-stress coefficient* α is defined as

$$\alpha \;=\; 1 - \frac{\mathsf{k_{fr}}}{\mathsf{k_s}}\,, \tag{2.4}$$

and the *fluid-solid coupling modulus* M as

$$\frac{1}{M} \;=\; \frac{\alpha}{\mathsf{k_s}} + \phi\left(\frac{1}{\mathsf{k_f}} - \frac{1}{\mathsf{k_s}}\right) . \tag{2.5}$$

With the physical assumption that

$$\mathsf{k_s} \;>\; \mathsf{k_{fr}} \quad , \quad \mathsf{k_s} \;>\; \mathsf{k_f}\,, \tag{2.6}$$

we have

$$\alpha > 0 \quad , \quad M > 0 \,. \tag{2.7}$$

Using expressions 2.3, we obtain the relation between k_G and k_{fr}, *cf.* [8, (4)], where k_{fr} is noted k_{m}:

$$\mathrm{k}_G = \lambda_G + \tfrac{2}{3}\mu_G = \lambda_G + \tfrac{2}{3}\mu_{\mathrm{fr}} \overset{2.3}{=} \lambda_G - \lambda_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} = \mathrm{k}_{\mathrm{fr}} + \alpha^2 M \,.$$

We list the physical parameters of the specific porous media in consideration in this report in Table 1. The media are filled with brine, which is inviscid in the case of shale and sandstone materials.

Physical parameters	Sandstone	Sand 1	Shale	Sand 2
Porosity ϕ (%)	0.2	0.3	0.16	0.3
Fluid Density ρ_{f} $(kg.L^{-1})$	1.04	1	1.04	1
Solid Density ρ_{s} $(kg.L^{-1})$	2.5	2.6	2.21	2.7
Viscosity η (mPa.s)	0	1	0	1
Permeability κ_0 (μm^2)	60	10	10	10
Tortuosity α_∞	2	3	2	3
Solid Bulk Modulus k_{s} (GPa)	40	35	7.6	36
Fluid Bulk Modulus k_{f} (GPa)	2.5	2.2	2.5	2.2
Frame Bulk Modulus k_{fr} (GPa)	20	0.4	6.6	7
Frame Shear Modulus μ_{fr} (GPa)	12	0.5	3.96	5

Table 1: Summary of the physical parameters of media in consideration in this report. The parameters for sand 1 are obtained from [19][Table 1], those for sandstone and shale from [11][Table 5], for sand 2 from [20][Table 1]. For the tests in sections 7 and after, we will use these materials. However, we will vary some of the parameters to highlight their effect on the solution.

In (2.8), we compare our notations with the ones used by Pride in formula [33, (9.15),(9.19)].

	Pride's notations	Our notation
Undrained bulk modulus	K_U	$\mathrm{k}_{\mathrm{fr}} + \alpha^2 M = \lambda_{\mathrm{fr}} + \frac{2}{3}\mu_{\mathrm{fr}} + \alpha^2 M = H - \frac{4}{3}\mu_{\mathrm{fr}}$
Undrained shear modulus	G	μ_{fr}
Biot incompressibilites	C	αM
constants[33, (9.18)]	$H = K_U + \frac{4}{3}G$	$\lambda_{\mathrm{fr}} + \frac{2}{3}\mu_{\mathrm{fr}} + \alpha^2 M + \frac{4}{3}\mu_{\mathrm{fr}} = \lambda_{\mathrm{fr}} + 2\mu_{\mathrm{fr}} + \alpha^2 M$

(2.8)

3 Equations

In poroelastic equations, in addition to the nine unknowns already existing in elastic equation (with six for the stress tensor and three for the displacement of the particle), there are new quantities due to the presence of pore structure and fluid. These are the *pore pressure* p, and the three components of the *displacement of fluid relative to the solid displacement* w, *cf.* [9]. In this report we will mainly work in frequency domain, with the *pulsation* ω, and with the following unknowns:

u frame displacement in frequency-domain formulation,

w relative fluid displacement in frequency-domain formulation,

p fluid pressure,

$\boldsymbol{\tau}$ stress tensor.

For the purpose of introducing the equation of motion in time, we introduce briefly the time-dependent quantities,

u frame displacement in time-domain formulation,

$\mathtt{w}$ relative fluid displacement in time-domain formulation.

There are also corresponding quantities for the fluid pressure and the stress tensor.

Remark 1 (Convention of ∂_t). We introduce the notion of convention of the time derivative.

- **Convention 1** This convention follows the one of Pride and uses $\partial_t \to -\mathrm{i}\omega$. This is also employed in Dupuy [15]. Here, the time-harmonic part is represented by $e^{-\mathrm{i}\omega t}$, which is equivalent to using the Fourier transform convention

$$\mathcal{F}_1 g \ := \ \int e^{\mathrm{i}\omega t} g(t)\, dt \quad \Rightarrow \quad \mathcal{F}\dot{g} = -\mathrm{i}\omega\, \mathcal{F} g\,. \tag{3.1}$$

A plane wave is given by a multiple of

$$e^{-\mathrm{i}\omega t}\ e^{\mathbf{i}\mathbf{k}\cdot\boldsymbol{x}}\,\mathbf{d} \quad \text{, with } \mathbf{d} \text{ the polarization.} \tag{3.2}$$

- **Convention 2** : In this convention, one takes $\partial_t \to \mathrm{i}\omega$. The time-harmonic part is thus represented by $e^{\mathrm{i}\omega t}$, and is equivalent to using the Fourier transform convention,

$$\mathcal{F}_2 g \ := \ \int e^{-\mathrm{i}\omega t} g(t)\, dt \quad \Rightarrow \quad \mathcal{F}\dot{g} = \mathrm{i}\omega\, \mathcal{F} g\,.$$

A plane wave is given by a multiple of

$$e^{\mathrm{i}\omega t}\, e^{\mathbf{i}\mathbf{k}\cdot\boldsymbol{x}}\,\mathbf{d}. \tag{3.3}$$

This form of plane wave was used in [17, Eqn 5.2.18]. △

3.1 Equations of motion

We describe the equations of motion both in time and frequency domain, using the definition of two conventions for the time derivative in frequency domain. We next study two approximation models for low-frequency and vanishing viscosity.

3.1.1 Equations of motion in time domain

The first equation of motion comes from balancing forces acting on each sample [33],

$$\nabla\cdot\boldsymbol{\tau} \ + \ \mathfrak{f}_{\mathrm{u}} \ = \ \rho_a\,\ddot{\mathrm{u}} \ + \ \rho_{\mathrm{f}}\ddot{\mathtt{w}}\,, \tag{3.4}$$

while the second one is a generalized Darcy's law that takes into account the dependence of the drag force (due to the viscosity of the fluid) on the frequency,

$$-\nabla \mathrm{p} \ + \ \mathfrak{f}_{\mathrm{w}} \ = \ \rho_{\mathrm{f}}\ddot{\mathrm{u}} \ + \ \mathcal{V}(t)\star\dot{\mathtt{w}}. \tag{3.5}$$

In the above equation, $\mathfrak{f}_{\mathrm{w}}$ and $\mathfrak{f}_{\mathrm{u}}$ are external volume forces, and drag operator $\mathcal{V}$ is defined such that,

$$\mathcal{V}(t)\star g \ := \ \mathcal{F}_1^{-1}\left(\frac{\eta}{k(\omega)}\ \mathcal{F}_1 g\right)\,. \tag{3.6}$$

In the definition of $\mathcal{V}$, $\mathcal{F}_1$ is the Fourier transform defined in (3.1) in convention 1 in t (see remark 1), and $k(\omega)$ is Pride's dynamic permeability, *cf.* [32, Eqn. 236],

$$\frac{1}{k(\omega)} = \frac{1}{k_0}\left(\sqrt{1-\mathrm{i}\frac{4}{m}\frac{\omega}{\omega_t}} - \mathrm{i}\frac{\omega}{\omega_t}\right)\,. \tag{3.7}$$

Here, the (dimensionless) number m and the transition frequency ω_t are defined as

$$m := \frac{\phi}{\alpha_\infty k_0}\lambda_{\mathrm{fr}}^2 \quad , \quad \omega_t := \frac{\phi}{\alpha_\infty k_0}\frac{\eta}{\rho_{\mathrm{f}}}\,. \tag{3.8}$$

The frequency ω_t separates the low-frequency viscous-flow behavior from the high-frequency inertial flow. The constant m is usually determined by experimental means with

$$4 \leq m \leq 8\,.$$

We have also denoted by $\sqrt{\bullet}$ the square root branch that uses the Principal value, i.e. for $z\in\mathbb{C}\setminus\{0\}$,

$$\sqrt{z} = \sqrt{|z|}\,e^{\mathrm{i}\,\mathrm{Arg}(z)/2} \quad , \quad \mathrm{Arg}\,z \in (-\pi,\pi]\,. \tag{3.9}$$

Thus $\mathrm{Re}\sqrt{} > 0$, while $\mathrm{Im}\sqrt{}$ can be positive or negative.

3.1.2 Equations of motion in frequency domain

As mentioned in the introduction, in the report, we work in the frequency domain, i.e. with responses due to time-harmonic disturbances. The drag force described by operator $\mathcal{V}$ has simpler expression in this case. In particular, since

$$(\mathcal{F}_2\mathcal{V})(w) \;=\; (\mathcal{F}_1\mathcal{V})(-w) \;=\; \frac{\eta}{k(-\omega)} = \frac{\eta}{k_0}\left(\sqrt{1+\mathrm{i}\frac{4}{m}\frac{\omega}{\omega_t}} + \mathrm{i}\frac{\omega}{\omega_t}\right),$$

the contribution of the drag force is now written as,

$$\mathcal{F}_1\left(\mathcal{V}(t)\star\dot{\mathtt{w}}\right) \;=\; (\mathcal{F}_1\mathcal{V})(\omega)\,(\mathcal{F}_1\dot{\mathtt{w}})(\omega) \;=\; \frac{\eta}{k(\omega)}\big(-\mathrm{i}\omega\big)\,\big(\mathcal{F}_1\mathtt{w}\big)\,,$$

$$\mathcal{F}_2\left(\mathcal{V}(t)\star\dot{\boldsymbol{\omega}}\right) \;=\; (\mathcal{F}_2\mathcal{V})(w)\,(\mathcal{F}_2\dot{\mathtt{w}})(\omega) = \frac{\eta}{k(-\omega)}\,\big(\mathrm{i}\omega\big)\,\big(\mathcal{F}_2\mathtt{w}\big)\,.$$

Equations using convention 1 In this convention $\partial_t \to -\mathrm{i}\omega$. The first equation of motion is formally transformed to

$$\nabla\cdot\boldsymbol{\tau} + \mathbf{f}_{\mathrm{u}} \;=\; -\omega^2\rho_a\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{f}}\mathrm{w}\,,$$

while the second one becomes

$$-\nabla\mathrm{p} + \mathbf{f}_{\mathrm{w}} \;=\; -\omega^2\rho_{\mathrm{f}}\mathrm{u} - \omega^2\left(\mathrm{i}\frac{\eta}{\omega\,k(\omega)}\right)\mathrm{w}\quad. \tag{3.10}$$

Here, $\mathbf{f}_{\mathrm{u}}$ and $\mathbf{f}_{\mathrm{w}}$ are time-harmonic external volume forces. As in [34, Eqn. 77], define

$$\tilde{\rho}(\omega) \;:=\; \mathrm{i}\frac{\eta}{\omega\,k(\omega)} \;=\; \mathrm{i}\frac{\eta}{\omega\,k_0}\left(\sqrt{1-\mathrm{i}\frac{4}{m}\frac{\omega}{\omega_t}} - \mathrm{i}\frac{\omega}{\omega_t}\right). \tag{3.11}$$

The equations of motion in frequency domain with Convention 1 are given as (see also [33, Eqn 9.8, 9.29]),

$$\textbf{Convention 1}\qquad \begin{aligned}\nabla\cdot\boldsymbol{\tau} + \mathbf{f}_{\mathrm{u}} \;&=\; -\omega^2\,\rho_a\,\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{f}}\,\mathrm{w}\,,\\ -\nabla\mathrm{p} + \mathbf{f}_{\mathrm{w}} \;&=\; -\omega^2\rho_{\mathrm{f}}\mathrm{u} - \omega^2\tilde{\rho}(\omega)\,\mathrm{w}\quad.\end{aligned} \tag{3.12}$$

Equations using convention 2 In this convention $\partial_t \to \mathrm{i}\omega$, thus the second equation of motion is given by

$$-\nabla\mathrm{p} + \mathbf{f}_{\mathrm{w}} \;=\; -\omega^2\rho_{\mathrm{f}}\mathrm{u} + \mathrm{i}\omega\frac{\eta}{k(-\omega)}\,\mathrm{w} \;=\; -\omega^2\rho_{\mathrm{f}}\mathrm{u} \;-\; \omega^2\left(-\mathrm{i}\frac{\eta}{\omega\,k(-\omega)}\right)\mathrm{w}\,.$$

Following [35, Eqn 9–10], define

$$\tilde{\tilde{\rho}}(\omega) \;:=\; -\mathrm{i}\frac{\eta}{\omega\,k(-\omega)} \;=\; -\mathrm{i}\frac{\eta}{\omega\,k_0}\left(\sqrt{1+\mathrm{i}\frac{4}{m}\frac{\omega}{\omega_t}} + \mathrm{i}\frac{\omega}{\omega_t}\right). \tag{3.13}$$

The equations of motion in frequency domain with Convention 2 are given as

$$\textbf{Convention 2}\qquad \begin{aligned}\nabla\cdot\boldsymbol{\tau} + \mathbf{f}_{\mathrm{u}} \;&=\; -\omega^2\rho_a\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{f}}\mathrm{w}\,,\\ -\nabla\mathrm{p} + \mathbf{f}_{\mathrm{w}} \;&=\; -\omega^2\rho_{\mathrm{f}}\mathrm{u} \;-\; \omega^2\,\tilde{\tilde{\rho}}(\omega)\,\mathrm{w}\quad.\end{aligned} \tag{3.14}$$

We unify both conventions by writing

$$\partial_t \;\to\; \mathfrak{s}\,\mathrm{i}\,\omega \quad\text{with}\quad \mathfrak{s} \;=\; \begin{cases}-1\quad, & \text{convention 1}\\ 1\quad, & \text{convention 2}\end{cases}.$$

We introduce the *dynamic density* as

$$\rho_{\mathrm{dyn}}(\omega) = \begin{cases}\tilde{\rho}(\omega)\quad, & \text{convention 1}\\ \tilde{\tilde{\rho}}(\omega)\quad, & \text{convention 2}\end{cases}. \tag{3.15}$$

The equations of motion in the frequency domain for both convention are

$$\boxed{\begin{aligned}\nabla\cdot\boldsymbol{\tau} + \mathbf{f}_{\mathrm{u}} \;&=\; -\omega^2\rho_a\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{f}}\mathrm{w}\,,\\ -\nabla\mathrm{p} + \mathbf{f}_{\mathrm{w}} \;&=\; -\omega^2\rho_{\mathrm{f}}\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{dyn}}(\omega)\,\mathrm{w}\quad.\end{aligned}} \tag{3.16}$$

Property of ρ_{dyn} We have

$$\mathrm{Re}\,\frac{\tilde{\tilde{\rho}}}{-\mathrm{i}} > 0 \quad , \quad \mathrm{Im}\,\frac{\tilde{\tilde{\rho}}}{-\mathrm{i}} \;>\; 0\,.$$

As a result of this,

$$\mathrm{Re}\,\tilde{\tilde{\rho}} > 0 \quad , \quad \mathrm{Im}\,\tilde{\tilde{\rho}} < 0\,. \tag{3.17}$$

We also have

$$\boxed{\tilde{\rho}(\omega) \;=\; \overline{\tilde{\tilde{\rho}}(\omega)}\,.} \tag{3.18}$$

In short,

$$\boxed{\mathrm{Re}\,\rho_{\mathrm{dyn}} > 0 \quad , \quad \mathrm{Im}\,\rho_{\mathrm{dyn}} \begin{cases} > 0 & \text{in Convention 1} \\ < 0 & \text{in Convention 2} \end{cases} .} \tag{3.19}$$

3.1.3 Low-frequency approximation

The derivation of the limiting form of (3.16) when taking $\omega \to 0$ while other parameters (including η) fixed is given in Appendix B. We replace the dynamic density ρ_{dyn} (3.15) by its limit $\rho_{\mathrm{dyn}}^{\mathrm{LF}}$ at $\omega \to 0$,

$$\rho_{\mathrm{dyn}}^{\mathrm{LF}} \;:=\; \rho_{\mathrm{w}} \;+\; \mathfrak{s}\,\mathrm{i}\frac{\eta}{\omega\,k_0} \;=\; \mathrm{i}\frac{\eta}{\omega}\left(-\mathrm{i}\omega\,\frac{\rho_{\mathrm{w}}}{\eta} \;+\; \mathfrak{s}\,k_0\right)\,. \tag{3.20}$$

In tis case, the equation at low-frequency is given by

$$\textbf{Low-freq} \qquad \begin{aligned} \nabla\cdot\boldsymbol{\tau} + \mathbf{f}_{\mathrm{u}} \;&=\; -\omega^2\,\rho_a\,\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{f}}\,\mathrm{w}\,, \\ -\nabla\mathrm{p} + \mathbf{f}_{\mathrm{w}} \;&=\; -\omega^2\rho_{\mathrm{f}}\,\mathrm{u} - \omega^2\,\rho_{\mathrm{dyn}}^{\mathrm{LF}}\,\mathrm{w}\,. \end{aligned} \tag{3.21}$$

3.1.4 Formal zero-viscosity limiting for a fixed positive frequency

Below, we write out the form of equation (3.16) when $\eta \to 0$ at a fixed frequency ω and with other parameters fixed. Note that $\omega_t \to 0$ when $\eta \to 0$, and ω_t is in the denominator of the definition of ρ_{dyn}. However, this has finite limit as $\eta \to 0$. It suffices to consider the calculation in Convention 1. Using the definition $\omega_t := \dfrac{\phi}{\alpha_\infty k_0}\dfrac{\eta}{\rho_{\mathrm{f}}}$, we write

$$\frac{\eta}{k(\omega)} = \frac{1}{k_0}\left(\sqrt{\eta^2 - \mathrm{i}\,\frac{4}{m}\,\omega\,\eta\,\frac{\eta}{\omega_t}} - \mathrm{i}\frac{\eta}{\omega_t}\omega\right) \;=\; \frac{1}{k_0}\left(\sqrt{\eta^2 - \frac{4\mathrm{i}}{m}\omega\eta\frac{\alpha_\infty k_0\rho_{\mathrm{f}}}{\phi}} - \mathrm{i}\omega\frac{\alpha_\infty k_0\rho_{\mathrm{f}}}{\phi}\right)\,.$$

For a fixed $\omega > 0$, compute the limit of $\frac{\eta}{k(\omega)}$ as $\eta \to 0$ (under the assumption that the quantities $\phi, \alpha_\infty, k_0, \Gamma, m$ are independent of η), and replace the expression of the dynamis density in (3.11). We obtain similar results for Convention 2. We note by $\rho_{\mathrm{dyn}}^{\mathrm{VV}}$ the vanishing viscosity limit of ρ_{dyn}, i.e.

$$\rho_{\mathrm{dyn}}^{\mathrm{VV}} \;:=\; \lim_{\substack{\eta\to 0,\\ \text{fixed }\omega>0}} \rho_{\mathrm{dyn}}(\omega) \;=\; \frac{\rho_{\mathrm{f}}}{\phi}\,\alpha_\infty\,. \tag{3.22}$$

We will use this expression in the numerical tests. For material with zero viscosity, we apply the following limiting form of (3.16),

$$\begin{matrix}\textbf{Vanishing} \\ \textbf{Viscosity}\end{matrix} \qquad \begin{aligned} \nabla\cdot\boldsymbol{\tau} + \mathbf{f}_{\mathrm{u}} \;&=\; -\omega^2\,\rho_a\,\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{f}}\,\mathrm{w}\,, \\ -\nabla\mathrm{p} + \mathbf{f}_{\mathrm{w}} \;&=\; -\omega^2\rho_{\mathrm{f}}\,\mathrm{u} - \omega^2\,\rho_{\mathrm{dyn}}^{\mathrm{VV}}\,\mathrm{w}\,. \end{aligned} \tag{3.23}$$

Remark 2. In Appendix B.2, we also derive a set of equations for vanishing viscosity starting with the equation in low-frequency (3.21). The equations have the same form, however with different values of dynamic density. The vanishing viscosity equation (3.23) uses $\rho_{\mathrm{dyn}}^{\mathrm{VV}}$ (3.22), while the vanishing viscosity at low-frequency (B.8) uses ρ_{w} defined in (B.7), with

$$\rho_{\mathrm{w}} \;:=\; \lim_{\substack{\eta\to 0,\\ \text{fixed }\omega>0}} \rho_{\mathrm{dyn}}^{\mathrm{LF}}(\omega) \;=\; \frac{\rho_{\mathrm{f}}}{\phi}\alpha_\infty\left(\frac{2}{m} + 1\right)\,.$$

However, in both cases, we can say that

$$\rho_{\text{dyn}}^{\text{VV}} \quad \text{and} \quad \rho_{\mathsf{w}} \quad \text{are real, positive and } \frac{\rho_{\text{f}}}{\phi} \times \text{ is a multiple of } \alpha_\infty.$$

We also recall that there is a 'formalism' in considering the limit of $\rho_{\text{dyn}}^{\text{LF}}$ in discussion of Appendix B.2, *cf.* (B.7). For a rigorous limit, we should have

$$\lim_{\substack{\eta\to 0,\\ \omega<\omega_t}} \omega\,\rho_{\text{dyn}}(\omega) \;=\; 0\,.$$

This means the quantity $\frac{\eta}{\omega k(\omega)}$ does not have uniform limit as $\eta \to 0$, and the limiting values depend on the ratio $\frac{\omega}{\eta}$. △

3.2 Constitutive laws

The first constitutive law is generalized from that of linear elasticity, taking into consideration the additional influence of fluid pressure,

$$\boldsymbol{\tau} \;=\; \mathbf{C}_{\text{fr}} \;:\; \boldsymbol{\epsilon}_{\text{fr}} \;-\; \alpha\,\mathrm{p}\,. \tag{3.24}$$

Here $\mathbf{C}_{\text{fr}}$ is the elastic stiff tensor of the drained frame, and $\boldsymbol{\epsilon}_{\text{fr}}$ is the strain tensor of the solid frame,

$$\boldsymbol{\epsilon} \;=\; \boldsymbol{\epsilon}_{\text{fr}} \;:=\; \frac{\nabla \mathrm{u} + (\nabla \mathrm{u})^t}{2}\,.$$

Under assumption of isotropy in the material making up the solid frame, the fluid and the frame, (3.24) reduces to

$$\boxed{\begin{aligned}\boldsymbol{\tau} \;&=\; \lambda_{\text{fr}} \nabla\cdot\mathrm{u} \;+\; 2\mu_{\text{fr}}\boldsymbol{\epsilon} \;-\; \alpha \mathrm{p}\\ &=\; 2\underbrace{\mu_{\text{fr}}}_{G}\boldsymbol{\epsilon} + \left(-\tfrac{2}{3}\mu_{\text{fr}} \;+\; \underbrace{\mathrm{k}_{\text{fr}} + M\alpha^2}_{\mathrm{k}_G \text{ in } [32]}\right) \nabla\cdot\mathrm{u}\,\mathbf{Id} \;+\; \alpha M \nabla\cdot\mathrm{w}\;\mathbf{Id}\\ &=\; 2\mu_{\text{fr}}\boldsymbol{\epsilon} \;+\; (\lambda_{\text{fr}} \;+\; M\alpha^2)\nabla\cdot\mathrm{u}\;\mathbf{Id} \;+\; \alpha M \nabla\cdot\boldsymbol{w}\;\mathbf{Id}\,.\end{aligned}} \tag{3.25}$$

The second constitutive law is

$$\boxed{\mathrm{p} \;=\; -M\,(\nabla\cdot\mathrm{w} + \mathbf{f}_{\mathrm{p}}) \;-\; M\,\alpha\nabla\cdot\mathrm{u}\,.} \tag{3.26}$$

Note that $\mathbf{f}_{\mathrm{p}}$ is a time-harmonic external source term. For more geophysics meaning of the above equations, we refer to the introduction of [33].

Remark 3. Here, the constitutive laws are expressed using the unknowns in the frequency domain. For those in time domain, we only have to replace the unknowns by the corresponding one in time domain, including the source in (3.26).

3.3 $\mathrm{u} - \mathrm{w}$ formulation

From now on, we consider the 2D problem in $\mathbb{R}^2$ in the (x, y)-plane. A time-harmonic planewave is diffracted by an infinitely long cylinder which has principal axis parallel to z-direction. In the 2D problem, the cylinder is approximated by its circular cross-section in (x, y)-plane. All the following vectors are in 2D, which means that the z component is equal to zero. We recall the following definitions and properties for scalar f and vector V:

$$\mathbf{curl}\,f \;=\; \begin{pmatrix}\partial_y f\\ -\partial_x f\end{pmatrix} \quad ; \quad \mathrm{curl}\,V \;=\; \partial_x V_y \;-\; \partial_y V_x\,,$$

$$\mathrm{div}\cdot\nabla = \Delta \quad , \quad \nabla\cdot(f\mathbf{Id}) = \nabla f \quad , \nabla\cdot\nabla^t V \;=\; \nabla\nabla\cdot V\,,$$

and

$$\Delta = -\mathbf{curl}\,\mathrm{curl} + \nabla\nabla\cdot\;.$$

Proposition 1 ($\mathrm{u}-\mathrm{w}$ formulation)**.** • *If* $(\mathrm{u},\mathrm{w},\boldsymbol{\tau},\mathrm{p})$ *solves the poroelastic system made up of* (3.16)*,* (3.25) *and* (3.26)*, then* (u,w) *solves the following system*

$$\begin{aligned} -\omega^2\rho_a\mathrm{u} \;-\; \rho_{\mathrm{f}}\omega^2\mathrm{w} \;-\; H\,\nabla\nabla\cdot\mathrm{u} \;+\; \mu_{\mathrm{fr}}\,\mathbf{curl}\,\mathrm{curl}\,\mathrm{u} \;-\; \alpha\,M\,\nabla\nabla\cdot\mathrm{w} = \mathbf{f}_{\mathrm{u}}\,,\\ -\omega^2\rho_{\mathrm{f}}\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{dyn}}(\omega)\,\mathrm{w} - M\,\nabla\nabla\cdot\mathrm{w} \;-\; M\alpha\nabla\nabla\cdot\mathrm{u} = \mathbf{f}_{\mathrm{w}} \;+\; \nabla M\,\mathbf{f}_{\mathrm{p}}\,. \end{aligned} \tag{3.27}$$

- *In reverse, if* (u,w) *solves* (3.27)*, with* $\boldsymbol{\tau}$ *and* p *given in terms of* u *and* w *by constitutive laws* (3.25) *and* (3.26)*, then* $(\mathrm{u},\mathrm{w},\boldsymbol{\tau},\mathrm{p})$ *solves the poroelastic system* (3.16)*.*

Proof. We need to express $\nabla\cdot\boldsymbol{\tau}$ and $\nabla\mathrm{p}$ in terms of divergence and curl of u and w. For $\nabla\mathrm{p}$, we have from (3.26)

$$-\nabla\mathrm{p} \;=\; M\,\nabla\nabla\cdot\mathrm{w} \;+\; M\,\nabla\mathbf{f}_{\mathrm{p}} \;+\; M\,\alpha\,\nabla\nabla\cdot\mathrm{u}\,,$$

and for $\nabla\cdot\boldsymbol{\tau}$, using (3.25), we have

$$\begin{aligned} \boldsymbol{\tau} \;&=\; 2\mu_{\mathrm{fr}}\,\boldsymbol{\epsilon} \;+\; (\lambda_{\mathrm{fr}} \;+\; M\alpha^2)\,\nabla\cdot\mathrm{u}\,\mathbf{Id} \;+\; \alpha\,M\,\nabla\cdot\mathrm{w}\,\mathbf{Id}\,;\\ \Rightarrow\quad \nabla\cdot\boldsymbol{\tau} \;&=\; \mu_{\mathrm{fr}}\,\nabla\cdot(\nabla+\nabla^t)\mathrm{u} \;+\; (\lambda_{\mathrm{fr}}+M\alpha^2)\nabla\nabla\cdot\mathrm{u} \;+\; \alpha\,M\,\nabla\nabla\cdot\mathrm{w}\\ &=\; \mu_{\mathrm{fr}}\,\Delta\mathrm{u} \;+\; (\mu_{\mathrm{fr}}+\lambda_{\mathrm{fr}}+M\alpha^2)\nabla\nabla\cdot\mathrm{u} \;+\; \alpha\,M\,\nabla\nabla\cdot\mathrm{w}\\ &=\; -\mu_{\mathrm{fr}}\,\mathbf{curl}\,\mathrm{curl}\,\mathrm{u} \;+\; \underbrace{(2\mu_{\mathrm{fr}} \;+\; \lambda_{\mathrm{fr}} \;+\; M\alpha^2)}_{:=H}\,\nabla\nabla\cdot\mathrm{u} \;+\; \alpha\,M\,\nabla\nabla\cdot\mathrm{w}\,. \end{aligned}$$

Here we have used the Biot coefficient H defined in (2.8),

$$H \;:=\; 2\mu_{\mathrm{fr}} \;+\; \lambda_{\mathrm{fr}} \;+\; M\,\alpha^2\,.$$

We next substitute these expressions into the equation of motions that are of the same form in both convention. The first equation

$$\nabla\cdot\boldsymbol{\tau} \;+\; \mathbf{f}_{\mathrm{u}} \;=\; -\omega^2\,\rho_a\,\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{f}}\,\mathrm{w}\,,$$

gives

$$-\mu_{\mathrm{fr}}\,\mathbf{curl}\,\mathrm{curl}\,\mathrm{u} \;+\; H\,\nabla\nabla\cdot\mathrm{u} \;+\; \alpha\mathfrak{m}\nabla\nabla\cdot\mathrm{w}+\mathbf{f}_{\mathrm{u}} \;=\; -\omega^2\,\rho_a\,\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{f}}\,\mathrm{w}\,.$$

The second one

$$-\nabla\mathrm{p} \;+\; \mathbf{f}_{\mathrm{w}} \;=\; -\omega^2\rho_{\mathrm{f}}\mathrm{u} \;-\; \omega^2\rho_{\mathrm{dyn}}(\omega)\,\mathrm{w}\,,$$

gives

$$M\nabla\nabla\cdot\mathrm{w} \;+\; M\nabla\mathbf{f}_{\mathrm{p}} \;+\; M\alpha\nabla\nabla\cdot\mathrm{u} \;+\; \mathbf{f}_{\mathrm{w}} \;=\; -\omega^2\,\rho_a\,\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{f}}\,\mathrm{w}\,.$$

The second direction is just obtained by rearrangement of the equations. □

3.4 First order formulation

In the first order formulation of the equations of motion, we work with the unknowns

$$\mathbf{u} \quad,\quad \mathbf{w} \quad,\quad \boldsymbol{\tau} \quad,\quad \mathrm{p}\,, \tag{3.28}$$

where

$$\mathbf{u} \,=\, \mathfrak{s}\,\mathrm{i}\,\omega\,\mathrm{u} \quad,\quad \mathbf{w} \,=\, \mathfrak{s}\,\mathrm{i}\,\omega\,\mathrm{u} \quad,\quad \tilde{\boldsymbol{\epsilon}} \,=\, \frac{\nabla\mathbf{u}+(\nabla\mathbf{u})^t}{2}\,. \tag{3.29}$$

$\mathbf{u}$ and $\mathbf{w}$ are interpreted as the time-harmonic solid velocity and the relative fluid velocity. They solve the system

$$\left\{\begin{aligned} &\nabla\cdot\boldsymbol{\tau}+\mathbf{f}_u && = \mathfrak{s}\,\mathrm{i}\,\omega\,\rho_a\,\mathbf{u} \;+\; \mathfrak{s}\,\mathrm{i}\,\omega\,\rho_{\mathrm{f}}\,\mathbf{w}\,,\\ &-\nabla\mathrm{p}+\mathbf{f}_{\mathrm{w}} && = \mathfrak{s}\,\mathrm{i}\,\omega\,\rho_{\mathrm{f}}\mathbf{u} \;+\; \mathfrak{s}\,\mathrm{i}\,\omega\;\rho_{\mathrm{dyn}}\mathbf{w}\,,\\ &\mathfrak{s}\,\mathrm{i}\,\omega\,\boldsymbol{\tau} && = \mathbf{C}_{\mathrm{fr}}:\tilde{\boldsymbol{\epsilon}} \,-\, \mathfrak{s}\,\mathrm{i}\,\omega\,\boldsymbol{\alpha}\,\mathrm{p}\,,\\ &\mathfrak{s}\,\mathrm{i}\,\omega\,(\mathrm{p}+M f_{\mathrm{p}}) && = -M\,\nabla\cdot\mathbf{w} \;-\; M\,\boldsymbol{\alpha}:\tilde{\boldsymbol{\epsilon}}\,. \end{aligned}\right. \tag{3.30}$$

Proof. We have

$$\mathfrak{s}\,\mathrm{i}\omega\,\mathrm{u} \;=\; \mathbf{u}\,, \quad \text{and}\; \mathfrak{s}\,\mathrm{i}\omega\,\mathrm{w} \;=\; \mathbf{w}\,.$$

We simply replace the relations above in the equations (3.16):

$$\nabla\cdot\boldsymbol{\tau} + \mathbf{f}_{\mathrm{u}} = (\mathfrak{s}\,\mathrm{i}\omega)^2\mathrm{u} \,-\, (\mathfrak{s}\,\mathrm{i}\omega)^2\,\mathrm{w} \quad\Longrightarrow\quad \nabla\cdot\boldsymbol{\tau} \,+\, \mathbf{f}_u \;=\; \mathfrak{s}\,\mathrm{i}\omega\,\rho_a\,\mathbf{u} \,+\, \mathfrak{s}\,\mathrm{i}\omega\,\rho_{\mathrm{f}}\,\mathbf{w}\,.$$

$$-\nabla\mathrm{p} + \mathbf{f}_{\mathrm{w}} \;=\; (\mathfrak{s}\,\mathrm{i}\omega)^2\,\rho_{\mathrm{f}}\,\mathrm{u} \,+\, (\mathfrak{s}\,\mathrm{i}\omega)^2\rho_{\mathrm{dyn}}\,\mathrm{w} \quad\Longrightarrow\quad -\nabla\mathrm{p} + \mathrm{p}_{\mathrm{w}} \;=\; \mathfrak{s}\,\mathrm{i}\omega\,\rho_{\mathrm{f}}\,\mathbf{u} \,+\, \mathfrak{s}\,\mathrm{i}\omega\,\rho_{\mathrm{dyn}}\,\mathbf{w}\,.$$

$$\boldsymbol{\tau} = \mathbf{C}_{\mathrm{fr}} : \frac{\nabla\mathrm{u} + (\nabla\mathrm{u})^t}{2} - \alpha\mathrm{p} \quad\Longrightarrow\quad \mathfrak{s}\,\mathrm{i}\omega\,\boldsymbol{\tau} \;=\; \mathbf{C}_{\mathrm{fr}} : \tilde{\boldsymbol{\epsilon}} \,-\, \mathfrak{s}\,\mathrm{i}\omega\,\boldsymbol{\alpha}\,\mathrm{p}\,.$$

$$\mathrm{p} \;=\; -\mathrm{m}\nabla\cdot\mathrm{w} \,-\, M f_{\mathrm{p}} \,-\, M\boldsymbol{\alpha} : \frac{\nabla\mathrm{u} + (\nabla\mathrm{u})^t}{2} \quad\Longrightarrow\quad \mathfrak{s}\,\mathrm{i}\omega\,\mathrm{p} \;=\; -M\,\nabla\cdot\mathbf{w} \,-\, \mathfrak{s}\,\mathrm{i}\omega M\,f_{\mathrm{p}} \,-\, M\,\boldsymbol{\alpha} : \tilde{\boldsymbol{\epsilon}}.$$

□

3.5 Boundary and interface conditions

We describe the conditions on the boundaries for three different configurations, first a bounded domain, then two interaction problems, a fluid-solid and porous-porous interaction.

Bounded domain Working with the unknowns of the first-order formulation (3.28), we can impose four types of boundary conditions on the boundaries of a bounded domain:

$$\text{Type 1} \qquad \begin{cases} \boldsymbol{\tau}\cdot\boldsymbol{n} &= \; f_t\,, \\ \mathbf{w}\cdot\boldsymbol{n} &= \; f_w\,, \end{cases} \tag{3.31}$$

$$\text{Type 2} \qquad \begin{cases} \boldsymbol{\tau}\cdot\boldsymbol{n} &= \; f_t\,, \\ \quad\mathrm{p} &= \; f_p\,, \end{cases} \tag{3.32}$$

$$\text{Type 3} \qquad \begin{cases} \quad\mathbf{u} &= \; f_u, \\ \quad\mathrm{p} &= \; f_p\,, \end{cases} \tag{3.33}$$

or

$$\text{Type 4} \qquad \begin{cases} \quad\mathbf{u} &= \; f_u\,, \\ \mathbf{w}\cdot\boldsymbol{n} &= \; f_w\,, \end{cases} \tag{3.34}$$

f_u, f_w, f_t, f_p being exterior forces, and $\boldsymbol{n}$ the normal vector along Γ pointing outward. Solution with these conditions are obtained in Section 7. These conditions are also used in the scattering of a plane wave by impenetrable obstacle, *cf.* Section 8. The free boundary conditions are a special case of (3.32),

$$\begin{cases} \boldsymbol{\tau}\cdot\boldsymbol{n} &= \; 0\,, \\ \quad\mathrm{p} &= \; 0\,. \end{cases} \qquad \text{Free boundary conditions.} \tag{3.35}$$

Interaction problems In interaction problems we will consider the reflection of a solid obstacle immersed in a solid or fluid infinite medium. Denote outer (infinite) medium by $\Omega_{(I)}$ and the solid obstacle by $\Omega_{(II)}$. Transmission conditions are imposed on the interface Γ between these two domains, i.e. on the boundary of the obstacle.

Porous-porous interaction problem When the outer medium is a poroelastic solid, the transmission conditions are, *cf.* section 9

$$\begin{cases} \mathbf{u}_{(I)} - \mathbf{u}_{(II)} = 0\,, \\ \mathrm{p}_{(I)} - \mathrm{p}_{(II)} = 0\,, \\ (\mathbf{w}_{(I)} - \mathbf{w}_{(II)})\cdot\boldsymbol{n} = 0\,, \\ (\boldsymbol{\tau}_{(I)} - \boldsymbol{\tau}_{(II)})\cdot\boldsymbol{n} = 0\,. \end{cases} \tag{3.36}$$

Fluid-porous interaction problem When the outer medium is a fluid, fluid-poroelastic transmission condition depends on the value of hydraulic permeability κ_Γ, *cf.* Section 10. We denote respectively by $\mathrm{p}_{\mathrm{flu}}$ and $\mathbf{u}_{\mathrm{flu}}$ the total pressure and velocity in the fluid. For a finite positive value of κ_Γ, we impose:

$$\begin{cases} (\mathbf{u}_{\mathrm{flu}} - \mathbf{u}) \cdot \boldsymbol{n} \,=\, \mathbf{w} \cdot \boldsymbol{n}\,, \\ \mathrm{p}_{\mathrm{flu}} - \mathrm{p} = \dfrac{1}{\kappa_\Gamma}\,\mathbf{w} \cdot \boldsymbol{n}\,, \\ \boldsymbol{\tau} \cdot \boldsymbol{n} = \,-\,\mathrm{p}_{\mathrm{flu}} \cdot \boldsymbol{n}\,, \end{cases} \tag{3.37}$$

where κ_Γ denotes the hydraulic permeability on the interface. In the fluid,

$$\mathbf{u}_{\mathrm{flu}} \,=\, -\,\frac{1}{\rho_{\mathrm{flu}}\,\mathfrak{s}\,\mathrm{i}\,\omega}\nabla\,\mathrm{p}_{\mathrm{flu}}\,. \tag{3.38}$$

We distinct extreme cases for κ_Γ: When $\kappa_\Gamma \to \infty$, the pores are *open*, and the second condition becomes $\mathrm{p}_{\mathrm{flu}} - \mathrm{p} = 0$. (3.37) becomes

$$\begin{cases} (\mathbf{u}_{\mathrm{flu}} - \mathbf{u}) \cdot \boldsymbol{n} \,=\, \mathbf{w} \cdot \boldsymbol{n}\,, \\ \mathrm{p}_{\mathrm{flu}} - \mathrm{p} = 0\,, \\ \boldsymbol{\tau} \cdot \boldsymbol{n} = \,-\,\mathrm{p}_{\mathrm{flu}} \cdot \boldsymbol{n}\,. \end{cases} \tag{3.39}$$

These equations are the ones used in [13]. On the other hand, when $\kappa_\Gamma = 0$, this is called *sealed pores*, and the second interface conditions is modified as $\mathbf{w} \cdot \boldsymbol{n} = 0$. (3.37) becomes

$$\begin{cases} (\mathbf{u}_{\mathrm{flu}} - \mathbf{u}) \cdot \boldsymbol{n} \,=\, \mathbf{w} \cdot \boldsymbol{n}\,, \\ \mathbf{w} \cdot \boldsymbol{n} \,=0, \\ \boldsymbol{\tau} \cdot \boldsymbol{n} = \,-\,\mathrm{p}_{\mathrm{flu}} \cdot \boldsymbol{n}\,. \end{cases} \qquad \text{or equivalently} \qquad \begin{cases} (\mathbf{u}_{\mathrm{flu}} - \mathbf{u}) \cdot \boldsymbol{n} \,=0, \\ \mathbf{w} \cdot \boldsymbol{n} \,=0, \\ \boldsymbol{\tau} \cdot \boldsymbol{n} = \,-\,\mathrm{p}_{\mathrm{flu}} \cdot \boldsymbol{n}\,. \end{cases} \tag{3.40}$$

Note that the subscript 'flu' indicates the unkwnowns in a fluid, while the subscript f denotes the unknowns and the parameters in the fluid contained in the pores of the poroelastic medium. The first and third equations in the equivalent form represent the perfect transmission in fluid-elastic scattering.[1]

4 Planewave Analysis

We are going to determine which forms of planewave are admissible solutions of (3.27) with zero sources. The analysis also gives the possible speeds of propagation sustained in a poroelastic medium. Here, we can observe a fast compressional wave and a shear wave as in elastic medium, but also a second slow compressional wave, associated physically to out-of-phase liquid and solid compressional particle motions.

With vectorial $\mathbf{k}$ and $\mathbf{d}$, a vectorial time-harmonic plane wave has the form

$$e^{\pm\mathrm{i}\omega t}\, e^{\pm\mathbf{k}\cdot\mathbf{x}}\,\mathbf{d}\,.$$

We will focus on the plane wave that attenuates along its propagation direction in a medium with viscosity. In particular, we consider slowness vector $\mathsf{s} = \mathsf{s}(\omega)$ satisfying

$$(-\mathfrak{s})\,\mathrm{Re}\,\mathsf{s} \,>\, 0 \quad , \quad \mathrm{Im}\,\mathsf{s} \,>\, 0\,,$$

(this is later defined in Definition (5.33)), and define wave vector $\mathbf{k}$ as

$$\mathbf{k} \,=\, \omega\,\mathsf{s}(\omega)\,\widehat{\mathbf{k}} \quad ,\text{with } |\widehat{\mathbf{k}}| \,=\, 1\,, \tag{4.1}$$

[1] In [1] the transmission condition between fluid and solid is

$$\boldsymbol{\tau} \cdot \boldsymbol{n} \,=\, \mathrm{p}_{\mathrm{flu}}\,,$$
$$\omega^2\,\rho_{\mathrm{flu}}\,\mathrm{u} \cdot \boldsymbol{n} \,=\, \nabla\mathrm{p}_{\mathrm{flu}} \cdot \boldsymbol{n}\,.$$

Here, the condition is in terms of solid displacement u. Since we work with the formulation of velocity, using the following identities,

$$\omega^2 = -(\mathfrak{s}\,\mathrm{i}\,\omega)^2\,, \qquad \mathbf{u} \,=\mathfrak{s}\,\mathrm{i}\omega\,\mathrm{u}\,, \qquad \nabla\mathrm{p}_{\mathrm{flu}} \,= -\rho_{\mathrm{flu}}\,\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u}_{\mathrm{flu}}\,.$$

we can write the second condition as:

$$-(\mathfrak{s}\,\mathrm{i}\,\omega)\,\rho_{\mathrm{flu}}\mathbf{u} \cdot \boldsymbol{n} \,=\, -\rho_{\mathrm{flu}}\,\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u}_{\mathrm{flu}} \cdot \boldsymbol{n} \quad\Rightarrow\quad \mathbf{u} \cdot \boldsymbol{n} \,=\, \mathbf{u}_{\mathrm{flu}} \cdot \boldsymbol{n}\,.$$

$\widehat{\mathbf{k}}$ denotes the direction of propagation. We consider the plane wave with polarization $\widehat{\mathbf{d}}$ with $|\widehat{\mathbf{d}}| = 1$,

$$e^{\mathrm{i}\,\mathsf{s}\,\omega t}\,e^{\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{x}}\,\mathbf{d}\,. \tag{4.2}$$

We rewrite the plane wave as,

$$e^{\omega\,\mathrm{i}\,\mathsf{s}\left(t\;-\;|\mathrm{Re}\,\mathsf{s}(\omega)|\,\widehat{\mathbf{k}}\cdot\boldsymbol{x}\right)}\,e^{-\,\omega\;\mathrm{Im}\,\mathsf{s}(\omega)\,\widehat{\mathbf{k}}\cdot\boldsymbol{x}}\qquad.$$

From this, we define the following physical quantities,

$$\begin{aligned}\text{Phase velocity}\qquad \mathsf{v}(\omega)\;&:=\;\frac{1}{|\mathrm{Re}\;\mathsf{s}(\omega)|}\,,\\ \text{Attenuation}\qquad \mathsf{a}(\omega)\;&:=\;\omega\;\mathrm{Im}\,\mathsf{s}(\omega)\,.\end{aligned} \tag{4.3}$$

We also work with the complex velocity,

$$\mathsf{c}(\omega)\;:=\;\frac{1}{\mathsf{s}(\omega)}\,. \tag{4.4}$$

4.1 Preliminary calculation

We will make use of the following identities with curl and **curl**.

- For $V = (V_x, V_y)$, we have the following product rules,

$$\begin{aligned}\mathrm{curl}(f\,V)\;&=\;\partial_x(f\,V_y)\;-\;\partial_y(f\,V_x)\;=\;(\partial_x f)\,V_y\;-\;(\partial_y f)\,V_x\;+\;f\,(\partial_x V_y\;-\;\partial_y V_x)\\ &=\;-(\mathbf{curl}\,f)\cdot V\;+\;f\,\mathrm{curl}\,V\,,\end{aligned} \tag{4.5}$$

  and

$$\mathbf{curl}(f\,g)\;=\;\begin{pmatrix}\partial_y(f\,g)\\ -\partial_x(f\,g)\end{pmatrix}\;=\;g\begin{pmatrix}\partial_y f\\ -\partial_x f\end{pmatrix}\;+\;f\begin{pmatrix}\partial_y g\\ -\partial_x g\end{pmatrix}\;=\;f\,\mathbf{curl}\,g\;+\;g\,\mathbf{curl}\,f\,.$$

- With $W = (W_x, W_y)$, if we define

$$V\times W\;:=\;V_1W_2 - V_2W_1\;=\;V\cdot\mathcal{R}_{-\frac{\pi}{2}}W\,,$$

  where

$$\mathcal{R}_{-\frac{\pi}{2}}W\;:=\;\begin{pmatrix}W_y\\ -W_x\end{pmatrix}\;=\;\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}W\,,$$

  we then have the equivalence

$$V\parallel W\quad\Leftrightarrow\quad V\times W = 0\,. \tag{4.6}$$

Lemma 2. *We have the following identities.*

$$\begin{aligned}\nabla\nabla\cdot e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\,\mathbf{d}\;&=\;-\,(\mathbf{d}\cdot\mathbf{k})\,\mathbf{k}\,e^{\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{x}}\,,\\ \mathbf{curl}\,\mathrm{curl}\;e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\mathbf{d}\;&=\;e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\Big(\mathbf{d}\times\mathbf{k}\Big)\,\mathcal{R}_{-\frac{\pi}{2}}\mathbf{k}\,.\end{aligned} \tag{4.7}$$

The right-hand-side of the second identity can also be written as,

$$(\mathbf{d}\times\mathbf{k})\,\mathcal{R}_{-\frac{\pi}{2}}\mathbf{k}\;=\;\Big(\mathbf{d}\cdot\mathcal{R}_{-\frac{\pi}{2}}\mathbf{k}\Big)\,\mathcal{R}_{-\frac{\pi}{2}}\mathbf{k}\;=\;|\mathbf{k}|^2\mathbf{d}\;-\;(\mathbf{k}\cdot\mathbf{d})\,\mathbf{k}\qquad. \tag{4.8}$$

Proof.

$$\begin{aligned}\nabla\cdot\left(e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\mathbf{d}\right)\;&=\;e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\;\mathbf{d}\cdot\nabla(\mathrm{i}\mathbf{k}\cdot\boldsymbol{x})\;=\;e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\;\mathbf{d}\cdot(\mathrm{i}\,\mathbf{k})\quad,\\ \nabla e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\;&=\;e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\,\nabla(\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}) = e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}(\mathrm{i}\mathbf{k})\,,\\ \Rightarrow\;\nabla\nabla\cdot e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\,\mathbf{d}\;&=\;-\,(\mathbf{d}\cdot\mathbf{k})\,\mathbf{k}\,e^{\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{x}}\,.\end{aligned}$$

We next consider the curl operator. By (4.5),

$$
\begin{aligned}
\operatorname{curl}\left(e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\,\mathbf{d}\right) \;&=\; -\left(\mathbf{curl}\,e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\right)\cdot\mathbf{d} \;=\; -\,\mathrm{i}\,e^{\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{x}}\,(\mathbf{d}\times\mathbf{k})\,,\\
\mathbf{curl}\,e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}} \;&=\; \mathrm{i}\,e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\,\mathbf{curl}(\mathbf{k}\cdot\boldsymbol{x}) \;=\; \mathrm{i}\,e^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\,\mathcal{R}_{-\frac{\pi}{2}}\mathbf{k}\,,\\
\Rightarrow \mathbf{curl}\,\operatorname{curl}e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\mathbf{d} \;&=\; \mathbf{curl}\Big(\,-\,\mathrm{i}\,e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\,\mathbf{d}\times\mathbf{k}\,\Big) \;=\; -\,\mathrm{i}\,\Big(\mathbf{d}\times\mathbf{k}\Big)\,\Big(\mathrm{i}e^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\,\mathcal{R}_{-\frac{\pi}{2}}\mathbf{k}\Big)\\
&=\; e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\Big(\mathbf{d}\times\mathbf{k}\Big)\,\mathcal{R}_{-\frac{\pi}{2}}\mathbf{k}\,.
\end{aligned}
$$

□

4.2 Admissible plane waves and slowness calculation

Proposition 3 (Plane wave solutions to (3.27))**.** *The three slownesses sustained in a poroelastic medium are*

$$
\textit{S-wave-slowness}\qquad \mathsf{s}_{\mathrm{S}}^2(\omega) \;:=\; \frac{\det A(\omega)}{\mu_{\mathrm{fr}}\,\rho_{\mathrm{dyn}}(\omega)}\,, \tag{4.9a}
$$

$$
\textit{`fast' P-wave-slowness}\qquad 2\,\mathsf{s}_{\mathrm{P}}^2(\omega) \;:=\; \frac{\operatorname{tr}C(\omega)}{\det B} \;-\; \sqrt{\left(\frac{\operatorname{tr}C(\omega)}{\det B}\right)^2 \;-\; 4\frac{\det A(\omega)}{\det B}}\,, \tag{4.9b}
$$

$$
\textit{`slow' P-wave-slowness}\qquad 2\,\mathsf{s}_{\mathrm{B}}^2(\omega) \;:=\; \frac{\operatorname{tr}C(\omega)}{\det B} + \sqrt{\left(\frac{\operatorname{tr}C(\omega)}{\det B}\right)^2 \;-\; 4\frac{\det A(\omega)}{\det B}}\,, \tag{4.9c}
$$

where we have defined

$$
A(\omega) := \begin{pmatrix}\rho_a & \rho_{\mathrm{f}}\\ \rho_{\mathrm{f}} & \rho_{\mathrm{dyn}}\end{pmatrix}\quad,\quad B := \begin{pmatrix}H & \alpha\,M\\ \alpha M & M\end{pmatrix}\quad,\quad B^{cof} = \begin{pmatrix}M & -\alpha\,M\\ -\alpha M & H\end{pmatrix}\quad,\quad C(\omega) := B^{cof}A(\omega)\,, \tag{4.10}
$$

and

$$
\begin{aligned}
\operatorname{tr}C(\omega) \;&=\; \rho_{\mathrm{dyn}}(\omega)\,H \;-\; 2\alpha\,M\,\rho_{\mathrm{f}} \;+\; \rho_a\,M\,,\\
\det B \;&=\; M\,H \;-\; (\alpha\,M)^2 \;=\; M\,(\lambda_{\mathrm{fr}} \;+\; 2\mu_{\mathrm{fr}})\,,\\
\det A(\omega) \;&=\; \rho_a\,\rho_{\mathrm{dyn}}(\omega) \;-\; \rho_{\mathrm{f}}^2\,.
\end{aligned} \tag{4.11}
$$

For $\bullet \in \{\mathrm{S}\,,\,\mathrm{P}\,,\,\mathrm{B}\}$*, if* $(\mathrm{u}_\bullet,\mathrm{w}_\bullet)$ *is of the form*

$$
\mathrm{u}_\bullet \;=\; e^{\mathrm{i}\,\mathbf{k}_\bullet\cdot\boldsymbol{x}}\,\widehat{\mathbf{d}}\quad,\quad \mathrm{w}_\bullet \;=\; \upbeta_\bullet\,e^{\mathrm{i}\,\mathbf{k}_\bullet\cdot\boldsymbol{x}}\,\widehat{\mathbf{d}}
$$

solving (3.27)*, then the slowness* $\mathsf{s}_\bullet$*, the polarization* $\widehat{\mathbf{d}}$ *and the direction of propagation* $\widehat{\mathbf{k}}$ *have to satisfy the following constraints.*

1. *The transverse plane wave (i.e. one with polarization direction perpendicular to the propagation direction) is given by the pair* $(\mathrm{u}_{\mathrm{S}},\mathrm{w}_{\mathrm{S}})$

$$
\begin{cases}
\mathbf{k}_{\mathrm{S}} \;=\; \omega\,\mathsf{s}_{\mathrm{S}}(\omega)\,\widehat{\mathbf{k}}\quad;\\
\mathsf{s}_{\mathrm{S}}(\omega)\ \textit{given by}\ (4.9\mathrm{a})\,,\\
\widehat{\mathbf{k}}\perp\ \widehat{\mathbf{d}}\quad,\quad |\widehat{\mathbf{k}}| = |\widehat{\mathbf{d}}| = 1\,,\\
\upbeta_{\mathrm{S}} \;=\; -\,\dfrac{\rho_{\mathrm{f}}}{\rho_{\mathrm{dyn}}(\omega)}\quad \textit{cf.}\,(4.19)\,.
\end{cases} \tag{4.12}
$$

2. *There are two types of longitudinal waves (i.e. those with polarization direction parallel to the propagation*

direction) given by the pair $(\mathrm{u}_\bullet, \mathrm{w}_\bullet)$ *with* $\bullet \in \{\mathrm{P}, \mathrm{B}\}$*, with*

$$\begin{cases} \mathbf{k}_\bullet \;=\; \omega\, \mathsf{s}_\bullet(\omega)\, \widehat{\mathbf{d}} \quad , \quad |\widehat{\mathbf{d}}| = 1, \\ \mathsf{s}_\bullet(\omega) \ \textit{given by (4.9b) or (4.9c)}, \\ \beta_\bullet \;=\; -\dfrac{H\, \mathsf{s}_\bullet^2(\omega) - \rho_a}{\alpha\, M\, \mathsf{s}_\bullet^2(\omega) - \rho_\mathrm{f}} \qquad \textit{cf.}\ (4.26)\,. \end{cases} \qquad (4.13)$$

Remark 4. In (4.12), we can choose unit vectors $\widehat{\mathbf{k}}$ and $\widehat{\mathbf{d}}$ as

$$\widehat{\mathbf{d}} = \mathcal{R}_{\pm\frac{\pi}{2}} \widehat{\mathbf{k}} \quad ; \quad \mathcal{R}_{-\pi/2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \mathcal{R}_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \qquad \triangle$$

Proof. **Step 1** We write the plane wave solutions as follows:

$$\mathrm{u} \;=\; u_0\, e^{\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{x}}\, \widehat{\mathbf{d}} \quad , \quad \mathrm{w} \;=\; w_0\, e^{\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{x}}\, \widehat{\mathbf{d}},$$

and we replace the expressions of $\nabla\nabla\cdot \mathrm{u}$ and $\mathbf{curl}\,\mathrm{curl}\,\mathrm{u}$ into (3.27). We then obtain

$$\begin{aligned} -\omega^2 \rho_a\, u_0\, \widehat{\mathbf{d}} \;-\; \rho_\mathrm{f}\, \omega^2\, w_0\, \widehat{\mathbf{d}} \;+\; H\, u_0\, (\widehat{\mathbf{d}}\cdot\mathbf{k})\, \mathbf{k} + \;\mu_\mathrm{fr}\, u_0\, (\widehat{\mathbf{d}}\times\mathbf{k})\, \mathcal{R}_{-\frac{\pi}{2}}\mathbf{k} \;+\; \alpha\, M\, w_0\, (\widehat{\mathbf{d}}\cdot\mathbf{k})\, \mathbf{k} = 0\,, \\ -\omega^2 \rho_\mathrm{f}\, u_0\, \widehat{\mathbf{d}} \;-\; \omega^2 \rho_\mathrm{dyn}\, w_0\, \widehat{\mathbf{d}} \;+\; M\, w_0\, (\widehat{\mathbf{d}}\cdot\mathbf{k})\, \mathbf{k} + M\alpha u_0\, (\widehat{\mathbf{d}}\cdot\mathbf{k})\, \mathbf{k} = 0\,. \end{aligned} \qquad (4.14)$$

Rearranging the terms in (4.14) by coefficients of u_0 and w_0, we have:

$$\begin{aligned} \Big(H\, u_0 \;+\; \alpha\, M\, w_0\Big)(\widehat{\mathbf{d}}\cdot\mathbf{k})\, \mathbf{k} \;-\; \omega^2\Big(\rho_a\, u_0 \;+\; \rho_\mathrm{f}\, w_0\Big)\, \widehat{\mathbf{d}} \;+\; \mu_\mathrm{fr}\, u_0\, (\widehat{\mathbf{d}}\times\mathbf{k})\, \mathcal{R}_{-\frac{\pi}{2}}\mathbf{k} \;=\; 0\,, \\ -\omega^2\Big(\rho_\mathrm{f} u_0 \;+\; \rho_\mathrm{dyn} w_0\Big)\, \widehat{\mathbf{d}} \;+\; M\, \Big(w_0 \;+\; \alpha\, u_0\Big)\, (\widehat{\mathbf{d}}\cdot\mathbf{k})\, \mathbf{k} \;=\; 0\,. \end{aligned} \qquad (4.15)$$

Step 2 - Tranverse plane waves A S plane wave has the property

$$\mathbf{k}\cdot\widehat{\mathbf{d}} \;=\; 0\,. \qquad (4.16)$$

(4.8) means, for S planewave,

$$(\widehat{\mathbf{d}}\times\mathbf{k})\, \mathcal{R}_{-\frac{\pi}{2}}\mathbf{k} \;=\; \mathbf{k}^2\, \widehat{\mathbf{d}}\,.$$

Using the above identity and dividing both equalities in (4.15) by $|\mathbf{k}|$, we obtain

$$\begin{aligned} -\mathsf{c}^2\, \Big(\rho_a\, u_0 \;+\; \rho_\mathrm{f}\, w_0\Big)\widehat{\mathbf{d}} + \mu_\mathrm{fr}\, u_0\, \widehat{\mathbf{d}} \;=\; 0\,, \\ -\mathsf{c}^2(\rho_\mathrm{f}\, u_0 \;+\; \rho_\mathrm{dyn}\, w_0)\, \widehat{\mathbf{d}} \;=\; 0 \quad . \end{aligned} \qquad (4.17)$$

Recall the inverse slowness $\mathsf{c} := \dfrac{\omega}{|\mathbf{k}|}$, defined in (4.4). The above system in matrix form is

$$\begin{pmatrix} \mathsf{c}^2\rho_a - \mu_\mathrm{fr} & \mathsf{c}^2\rho_\mathrm{f} \\ \mathsf{c}^2\rho_\mathrm{f} & \mathsf{c}^2\rho_\mathrm{dyn} \end{pmatrix} \begin{pmatrix} u_0 \\ w_0 \end{pmatrix} = 0\,.$$

This means, assuming that u_0, w_0 do not vanish, that the above matrix is not invertible and has zero determinant

$$\mathsf{c}^2\rho_\mathrm{dyn}\, (\mathsf{c}^2\rho_a - \mu_\mathrm{fr}) - \mathsf{c}^4\rho_\mathrm{f}^2 \;=\; 0 \quad \Leftrightarrow \quad \mathsf{c}^2\Big[\mathsf{c}^2\, \big(\rho_\mathrm{dyn}\, \rho_a - \rho_\mathrm{f}^2\big) \;-\; \mu_\mathrm{fr}\rho_\mathrm{dyn}\Big] \;=\; 0\,.$$

We define the nonzero root to be

$$\mathsf{c}_\mathrm{S}(\omega) \;=\; \left(\frac{\mu_\mathrm{fr}\, \rho_\mathrm{dyn}(\omega)}{\rho_\mathrm{dyn}(\omega)\, \rho_a \;-\; \rho_\mathrm{f}^2}\right)^{1/2}.$$

The associated slowness is then

$$\text{Shear-wave slowness} \qquad \mathsf{s}_\mathrm{S}(\omega) \;=\; \left(\frac{\rho_\mathrm{dyn}(\omega)\, \rho_a \;-\; \rho_\mathrm{f}^2}{\mu_\mathrm{fr}\, \rho_\mathrm{dyn}(\omega)}\right)^{1/2}. \qquad (4.18)$$

A corresponding eigenvector is

$$u_0 = 1 \quad , \quad w_0 \;=\; -\,\frac{\rho_{\rm f}}{\rho_{\rm dyn}(\omega)}\,. \tag{4.19}$$

The last equality comes from the definition of $\rho_{\rm dyn}$ in (3.11) and (3.15) . This agrees with [33, Eqn 9.16]

Step 3 - Longitudinal plane waves

- A P planewave has the property

$$\mathbf{k}\times\widehat{\mathbf{d}} \;=\; 0\,. \tag{4.20}$$

Since $\mathbf{k}$ is parallel to $\widehat{\mathbf{d}}$, we can write

$$\widehat{\mathbf{d}} = a\,\mathbf{k} \quad , \quad \widehat{\mathbf{d}}\cdot\mathbf{k} = a|\mathbf{k}|^2\,, \qquad a\in\mathbb{R}\,,$$

then equation (4.15) becomes

$$\begin{aligned} \Big(H u_0 + \alpha M w_0\Big)(\widehat{\mathbf{d}}\cdot\mathbf{k})\mathbf{k} - \omega^2\Big(\rho_a\, u_0 + \rho_{\rm f}\, w_0\Big)\,\widehat{\mathbf{d}} \;&=\; 0\,,\\ -\omega^2(\rho_{\rm f}\, u_0 \,+\, \rho_{\rm dyn}\, w_0)\,\widehat{\mathbf{d}} + M\Big(w_0 \,+\, \alpha\, u_0\Big)(\widehat{\mathbf{d}}\cdot\mathbf{k})\mathbf{k} \;&=\; 0\,. \end{aligned} \tag{4.21}$$

$$\begin{aligned} \Rightarrow\; \Big(H\, u_0 \,+\, \alpha\, M\, w_0\Big)\, a\,|\mathbf{k}|^2\mathbf{k} - \omega^2\Big(\rho_a\, u_0 \,+\, \rho_{\rm f}\, w_0\Big)\, a\,\mathbf{k} \;&=\; 0\,,\\ -\omega^2(\rho_{\rm f}\, u_0 \,+\, \rho_{\rm dyn}\, w_0)\, a\,\mathbf{k} + M\Big(w_0 \,+\, \alpha\, u_0\Big)\, a\,|\mathbf{k}|^2\mathbf{k} \;&=\; 0\,. \end{aligned} \tag{4.22}$$

Divide (4.22) by a and $|\mathbf{k}|^2$, gives

$$\begin{aligned} \Big(H\, u_0 + \,\alpha\, M\, w_0\Big) - \mathsf{c}^2\Big(\rho_a\, u_0 + \rho_{\rm f}\, w_0\Big) \;&=\; 0\,,\\ -\mathsf{c}^2(\rho_{\rm f}\, u_0 + \rho_{\rm dyn}\, w_0) + M\Big(w_0 \,+ \alpha\, u_0\Big) \;&=\; 0\,. \end{aligned} \tag{4.23}$$

Written in matrix form, we obtain

$$\begin{pmatrix} H - \mathsf{c}^2\rho_a & \alpha M - \mathsf{c}^2\rho_{\rm f}\\ -\mathsf{c}^2\rho_{\rm f} + \alpha M & -\mathsf{c}^2\rho_{\rm dyn} + M\end{pmatrix}\begin{pmatrix}u_0\\ w_0\end{pmatrix} = 0\,,$$

or

$$B\begin{pmatrix}u_0\\ w_0\end{pmatrix} - \mathsf{c}^2 A\begin{pmatrix}u_0\\ w_0\end{pmatrix} = 0 \quad\Rightarrow\quad C\begin{pmatrix}u_0\\ w_0\end{pmatrix} - (\det A)\mathsf{c}^2\begin{pmatrix}u_0\\ w_0\end{pmatrix} = 0\,,$$

where we have defined

$$B := \begin{pmatrix} H & \alpha M\\ \alpha M & M\end{pmatrix} \quad , \quad A := \begin{pmatrix}\rho_a & \rho_{\rm f}\\ \rho_{\rm f} & \rho_{\rm dyn}\end{pmatrix} \quad , A^{\rm cof} = \begin{pmatrix}\rho_{\rm dyn} & -\rho_{\rm f}\\ -\rho_{\rm f} & \rho_a\end{pmatrix} \quad ,$$

$$C := A^{\rm cof}\, B \;=\; \begin{pmatrix}\rho_{\rm dyn} & -\rho_{\rm f}\\ -\rho_{\rm f} & \rho_a\end{pmatrix}\begin{pmatrix} H & \alpha M\\ \alpha M & M\end{pmatrix} = \begin{pmatrix}\rho_{\rm dyn}\, H - \alpha M \rho_{\rm f} & \rho_{\rm dyn}\alpha M - \rho_{\rm f} M\\ -\rho_{\rm f} H + \rho_a \alpha M & -\rho_{\rm f}\alpha M + \rho_a M\end{pmatrix}.$$

This means $(\det A)\,\mathsf{c}^2$ is an eigenvalue of C. Note that both A and B are symmetric, thus diagonalizable.

We next consider the eigenvalues $\tilde{\mathsf{c}}$ of C, which satisfy the quadractic

$$\tilde c^2 - \tilde c\,{\rm tr}\, C + \det C \;\; = 0\,,$$

and are thus given by

$$2\tilde c := {\rm tr}\, C \mp \sqrt{({\rm tr}\, C)^2 \,-\, 4\det C}\,.$$

This means

$$2\mathsf{c}^2 = \frac{1}{\det A}\left({\rm tr}\, C \mp \sqrt{({\rm tr}\, C)^2 \,-\, 4\det C}\right)\,.$$

Since

$$\det C \;=\; (\det A^{\rm cof})\,\det B = (\det A)\,(\det B) \quad ,$$

we have

$$2\,\mathsf{c}^2 \;=\; \frac{{\rm tr}\, C}{\det A} \mp \sqrt{\left(\frac{{\rm tr}\, C}{\det A}\right)^2 - 4\frac{\det B}{\det A}}\,. \tag{4.24}$$

Longitudinal slowness

- Consider the slowness then $(\det B)\mathsf{s}$ is an eigenvalue of $\tilde{C} := B^{\mathrm{cof}}A$. We note that

$$\tilde{C} := \begin{pmatrix} M & -\alpha\, M \\ -\alpha M & H \end{pmatrix} \begin{pmatrix} \rho_a & \rho_{\mathrm{f}} \\ \rho_{\mathrm{f}} & \rho_{\mathrm{dyn}} \end{pmatrix} \quad , \quad \Rightarrow \mathrm{tr}\,\tilde{C} = M\rho_a - 2\alpha M \rho_{\mathrm{f}} + H\rho_{\mathrm{dyn}} \;=\; \mathrm{tr}\, C\,,$$

$$\det \tilde{C} = (\det B)(\det A)\,.$$

As a result,

$$\begin{aligned} &\text{'fast' P-wave-slowness} \quad , \quad 2\,\mathsf{s}_{\mathrm{P}}^2 \;:=\; \frac{\mathrm{tr}\, C}{\det B} - \sqrt{\left(\frac{\mathrm{tr}\, C}{\det B}\right)^2 - 4\frac{\det A}{\det B}}\,, \\ &\text{'slow' P-wave-slowness} \quad , \quad 2\,\mathsf{s}_{\mathrm{B}}^2 \;:=\; \frac{\mathrm{tr}\, C}{\det B} + \sqrt{\left(\frac{\mathrm{tr}\, C}{\det B}\right)^2 - 4\frac{\det A}{\det B}}\,. \end{aligned} \tag{4.25}$$

The components of corresponding eigenvectors are read from (4.23):

$$u_0 = 1 \quad , \quad w_0 \;=\; -\frac{H\,\mathsf{s}_\bullet^2 - \rho_a}{\mathsf{s}_\bullet^2\,\alpha\, M - \rho_{\mathrm{f}}}\,, \qquad \mathsf{s}_\bullet \in \{\mathsf{s}_{\mathrm{P}}\,, \mathsf{s}_{\mathrm{B}}\}\,. \tag{4.26}$$

□

Equivalence with Pride notations [33] We compare with the notations used in formula [33, (9.15),(9.19)] for the matrices B and C.

	Pride's notations	Our notation
Undrained bulk modulus	K_U	$\mathsf{k}_{\mathrm{fr}} + \alpha^2 M = \lambda_{\mathrm{fr}} + \frac{2}{3}\mu_{\mathrm{fr}} + \alpha^2 M = H - \frac{4}{3}\mu_{\mathrm{fr}}$
Undrained shear modulus	G	μ_{fr}
Biot incompressibilites constants[33, (9.18)]	C	αM
	M	M
	$H = K_U + \frac{4}{3}G$	$\lambda_{\mathrm{fr}} + \frac{2}{3}\mu_{\mathrm{fr}} + \alpha^2 M + \frac{4}{3}\mu_{\mathrm{fr}} \;= \lambda_{\mathrm{fr}} + 2\mu_{\mathrm{fr}} + \alpha^2 M = H$
	$HM - C^2$	$\det B = MH - (\alpha M)^2 \;=\; M(\lambda_{\mathrm{fr}} + 2\mu_{\mathrm{fr}})$
	$\rho_a H + \tilde{\rho} M - 2\rho_{\mathrm{f}} C$	$\mathrm{tr}\, C = \rho_{\mathrm{dyn}} H - 2\alpha M \rho_{\mathrm{f}} + \rho_a M$
[33, (9.20)]	$\gamma = \dfrac{\rho_a M + \rho_{\mathrm{dyn}} H - 2\rho_{\mathrm{f}} C}{HM - C^2}$	$\dfrac{\mathrm{tr}\, C}{\det B}$

(4.27)

4.3 First order formulation of the plane wave

We describe the plane wave solution using by the first-order formulation of the equations, then we present the expansion of the obtained plane wave using Bessel functions.

4.3.1 The corresponding plane wave solution

Using the slowness expresssions from (4.9), the plane wave writes:

1. For the transverse wave (polarization direction perpendicular to the propagation direction):

$$\mathbf{u}_{\mathrm{S}} \;=\; e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}}\;(\mathfrak{s}\,\mathrm{i}\,\omega)\;\widehat{\mathbf{d}} \quad , \quad \mathbf{w}_{\mathrm{S}} \;=\; \beta_{\mathrm{S}}\, e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}}\;(\mathfrak{s}\,\mathrm{i}\,\omega)\;\widehat{\mathbf{d}}\,, \tag{4.28a}$$

$$\boldsymbol{\tau}_{\mathrm{S}} \;=\; \mathrm{i}\,\omega\,\mathsf{s}_{\mathrm{S}}(\omega)\, e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}}\;\mu_{\mathrm{fr}}\left(\widehat{\mathbf{k}}\otimes\widehat{\mathbf{d}} \,+\, \widehat{\mathbf{d}}\otimes\widehat{\mathbf{k}}\right), \tag{4.28b}$$

$$\mathrm{p}_{\mathrm{S}} \;=\; 0\,. \tag{4.28c}$$

Physical parameters	Sandstone	Sand 1	Shale	Sand 2
Porosity ϕ (%)	0.2	0.3	0.16	0.3
Fluid Density ρ_{f} ($10^3 kg.m^{-3}$)	1.04	1	1.04	1
Solid Density ρ_{s} ($10^3 kg.m^{-3}$)	2.5	2.6	2.21	2.7
Viscosity η ($10^{-3} Pa.s$)	0	1	0	1
Permeability κ_0 ($10^{-9} m^2$)	0.06	0.01	0.01	0.01
Tortuosity α_∞	2	3	2	3
Solid Bulk Modulus $\mathrm{k_s}$ ($10^9 Pa$)	40	35	7.6	36
Fluid Bulk Modulus $\mathrm{k_f}$ ($10^9 Pa$)	2.5	2.2	2.5	2.2
Frame Bulk Modulus $\mathrm{k_{fr}}$ ($10^9 Pa$)	20	0.4	6.6	7
Frame Shear Modulus μ_{fr} ($10^9 Pa$)	12	0.5	3.96	5
Velocity ($m.s^{-1}$)				
v_P	4247	(1860, 4)	2481	(2866, 0.1)
v_B	1021	(82,70)	1127	(190,163)
v_S	2388	(486,1)	1429	(1512, 1)
Transition frequency	—	1kHz	—	1kHz
ρ_{dyn}	10.4	(12.5, -80)	13	(12.5, -80)
$\rho_a + \rho_{\mathrm{dyn}}$	12.6	(14.6, -80)	15	(14.7, -80)
$\mathrm{detA} = \rho_a \rho_{\mathrm{dyn}} - \rho_{\mathrm{f}}^2$	21.88	(25.5 -169)	25.2	(26.4, -174)
$H + M$	50.28	13.74	28.8	24.6
$\mathrm{detB} = H M - \alpha^2 M^2$	411.43	6.83	197.14	90.8
$\mathrm{trC} = \rho_{\mathrm{dyn}} H - 2\alpha M \rho_{\mathrm{f}} + \rho_a M$	417.46	(92.54, -583)	187.2	(228.6, -1432)
$4\,\dfrac{\mathrm{detA}}{\mathrm{detB}}$	0.212	(14.92, -99)	0.51	(1.16, -7.7)
$\left(\dfrac{\mathrm{trC}}{\mathrm{detB}}\right)^2$	1.03	(-7104, -2311)	0.9	(-242, -7.9)
$\left(\dfrac{\mathrm{trC}}{\mathrm{detB}}\right)^2 - 4\,\dfrac{\mathrm{detA}}{\mathrm{detB}}$	0.817	(-7119, -2212)	0.39	(-243,-72)
$\sqrt{\left(\dfrac{\mathrm{trC}}{\mathrm{detB}}\right)^2 - 4\,\dfrac{\mathrm{detA}}{\mathrm{detB}}}$	0.904	(12.96 , -85)	0.624	(2.27, -15.8)
$\dfrac{\mathrm{trC}}{\mathrm{detB}}$	1.015	(13.54, -85)	0.95	(2.51, -15.8)
$\mathsf{s}_{\mathrm{P}}^2$	5.54D-2	(0.29, -1.2D-3)	0.16	(0.12, -8D-5)
s_{P} ($10^{-3}\mathrm{s.m}^{-1}$)	0.235	(0.54, -1.1D-3)	0.4	(0.35, -1D-4)
$\mathsf{s}_{\mathrm{B}}^2$	0.96	(13.24, -85)	0.787	(2.39, -15.8)
s_{B} ($10^{-3}\mathrm{s.m}^{-1}$)	0.979	(7, -6)	0.89	(3, -2.6)
$\mathsf{s}_{\mathrm{S}}^2$	0.175	(4.24, -2.5D-2)	0.49	(0.44, -2.4D-3)
s_{S} ($10^{-3}\mathrm{s.m}^{-1}$)	0.419	(2.06, -6D-3)	0.7	(0.66, -1.8D-3)

Table 2: Summary of the physical parameters of media in consideration in this report. The parameters for sand 1 are obtained from [19][Table 1], those for sandstone and shale from [11][Table 5], for sand 2 from [20][Table 1]. Materials velocities and dynamic parameters are calculated for a frequency $f = 200$ Hz, and $\mathfrak{s} = -1$. The definition of slowness follows the equation (5.33).

with polarization given by

$$\begin{cases} \mathbf{k}_{\mathrm{S}} \;=\; \omega\, \mathsf{s}_{\mathrm{S}}(\omega)\, \widehat{\mathbf{k}} \quad , \\ \mathsf{s}_{\mathrm{S}}(\omega) \text{ given by } (4.9a)\,, \\ \widehat{\mathbf{k}} \perp \widehat{\mathbf{d}} \quad , \quad |\widehat{\mathbf{k}}| = |\widehat{\mathbf{d}}| = 1\,, \\ \beta_{\mathrm{S}} \;=\; \dfrac{\rho_{\mathrm{f}}}{\rho_{\mathrm{dyn}}(\omega)} \quad \textit{cf.}\,(4.19)\,. \end{cases} \tag{4.29}$$

2. For the two types of longitudinal waves P and B (polarization direction parallel to the propagation direction), which are distinguished by subscript $\bullet \in \{\mathrm{P}\,,\mathrm{B}\}$:

$$\mathbf{u}_\bullet \;=\; e^{\mathrm{i}\,\mathbf{k}_\bullet\cdot\boldsymbol{x}}\;(\mathfrak{s}\,\mathrm{i}\,\omega)\;\widehat{\mathbf{d}} \quad , \quad \mathbf{w}_\bullet \;=\; \upbeta_\bullet\, e^{\mathrm{i}\,\mathbf{k}_\bullet\cdot\boldsymbol{x}}\;(\mathfrak{s}\,\mathrm{i}\,\omega)\;\widehat{\mathbf{d}}\,, \tag{4.30a}$$

$$\boldsymbol{\tau}_\bullet \;=\; \mathrm{i}\,\omega\,\mathsf{s}_\bullet(\omega)\,e^{\mathrm{i}\mathbf{k}_\bullet\cdot\boldsymbol{x}}\left(2\,\mu_{\mathrm{fr}}\,\widehat{\mathbf{d}}\otimes\widehat{\mathbf{d}} + \Big(\underbrace{-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}}}_{\lambda_{\mathrm{fr}}} + M\,\alpha^2 \,+\, \upbeta_\bullet\,\alpha\,M\Big)\,\mathbf{Id}\right)\,, \tag{4.30b}$$

$$\mathrm{p}_\bullet \;=\; \mathrm{i}\,\omega\,\mathsf{s}_\bullet(\omega)\;(-M\,\upbeta_\bullet \;-\; M\,\alpha)\,e^{\mathrm{i}\mathbf{k}_\bullet\cdot\boldsymbol{x}}\,. \tag{4.30c}$$

with polarization given by

$$\begin{cases} \mathbf{k}_\bullet \;=\; \omega\,\mathsf{s}_\bullet(\omega)\,\widehat{\mathbf{d}} \quad , \quad |\widehat{\mathbf{d}}| = 1, \\ \mathsf{s}_\bullet(\omega) \text{ given by (4.9b) or (4.9c)}\,, \\ \upbeta_\bullet \;=\; -\dfrac{H\,\mathsf{s}_\bullet^2(\omega) - \rho_a}{\alpha\,M\,\mathsf{s}_\bullet^2(\omega) \,-\, \rho_{\mathrm{f}}} \qquad \textit{cf.}\ (4.26)\,. \end{cases} \tag{4.31}$$

Proof. The expression of the second order plane wave is of the form:

$$\mathrm{u}_\bullet \;=\; e^{\mathrm{i}\,\mathbf{k}_\bullet\cdot\boldsymbol{x}}\;\mathbf{d} \quad , \quad \mathrm{w}_\bullet \;=\; \upbeta_\bullet\,e^{\mathrm{i}\,\mathbf{k}_\bullet\cdot\boldsymbol{x}}\;\mathbf{d}\,, \quad \bullet \in \{\mathrm{S}\,,\,\mathrm{P}\,,\,\mathrm{B}\}\,.$$

The velocities are the time-derivative of the displacement. As a result, we obtain:

$$\mathbf{u}_\bullet \;=\; e^{\mathrm{i}\,\mathbf{k}_\bullet\cdot\boldsymbol{x}}\,(\mathfrak{s}\,\mathrm{i}\,\omega)\,\mathbf{d} \quad , \quad \mathbf{w}_\bullet \;=\; \upbeta_\bullet\,e^{\mathrm{i}\,\mathbf{k}_\bullet\cdot\boldsymbol{x}}\,(\mathfrak{s}\,\mathrm{i}\,\omega)\,\mathbf{d}\,, \quad \bullet \in \{\mathrm{S}\,,\,\mathrm{P}\,,\,\mathrm{B}\}\,.$$

The expression of the pressure p is given by (3.26). By using the relation $\nabla\cdot\;(e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\mathbf{d}) \;=\; \mathrm{i}\,e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\mathbf{d}\cdot\mathbf{k}$ and replacing the value of the plane wave, we obtain for p:

$$\begin{aligned} \mathrm{p}_\bullet \;&=\; -M\,\nabla\cdot\mathrm{w} \;-\; M\,\alpha\nabla\cdot\mathrm{u} \\ &=\; \mathrm{i}\,\mathbf{d}\cdot\mathbf{k}\,(-M\,\upbeta_\bullet \;-\; M\,\alpha)\,e^{\,\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\,. \end{aligned} \tag{4.32}$$

The stress tensor $\boldsymbol{\tau}$ is expressed in (3.25). We replace the value of $\nabla\cdot\mathrm{u}$, $\nabla\cdot\mathrm{w}$ and $\boldsymbol{\epsilon}_{\mathrm{fr}} \;=\; \mathrm{i}\,e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\frac{1}{2}\,(\mathbf{d}\otimes\mathbf{k}+\mathbf{k}\otimes\mathbf{d})$, to obtain:

$$\begin{aligned} \boldsymbol{\tau}_\bullet \;&=\; 2\mu_{\mathrm{fr}}\boldsymbol{\epsilon} \,+\, \Big(\lambda_{\mathrm{fr}} + M\alpha^2\Big)\,\nabla\cdot\mathrm{u}\;\mathbf{Id} \,+\, \alpha M\nabla\cdot\mathrm{w}\;\mathbf{Id} \\ &=\; \mathrm{i}\,e^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}\Big(\mu_{\mathrm{fr}}\,(\mathbf{k}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{k}) \,+\, \big(\lambda_{\mathrm{fr}} + M\alpha^2 \,+\, \upbeta_\bullet\,\alpha\,M\big)\;\mathbf{d}\cdot\mathbf{k}\;\mathbf{Id}\Big)\,. \end{aligned} \tag{4.33}$$

To finish the proof, we only need to symplify the expression of $\boldsymbol{\tau}$ and p by using $\mathbf{k}\cdot\mathbf{d} \,=\, 0$ for transverse wave and $\mathbf{k}\times\mathbf{d} \,=\, 0$ for longitudinal waves.

□

4.3.2 Expansion of the incident plane wave in Bessel functions

The incident plane wave is expanded to form a right-hand side vector.

For a longitudinal wave Recall the admissible longitudinal planewave allowed in an isotropic poroelastic medium from (4.30), $\bullet \in \{\mathrm{P}\,,\mathrm{B}\}$.

$$\mathbf{u}_\bullet^{\mathrm{pw}} \;=\; e^{\mathrm{i}\,\mathbf{k}_\bullet\cdot\boldsymbol{x}}\;(\mathfrak{s}\,\mathrm{i}\,\omega)\;\widehat{\mathbf{d}} \;=\; \frac{\mathfrak{s}}{\mathsf{s}_\bullet}\nabla(e^{\mathrm{i}\,\mathbf{k}_\bullet\cdot\boldsymbol{x}}) \quad , \tag{4.34a}$$

$$\mathbf{w}_\bullet^{\mathrm{pw}} \;=\; \upbeta_\bullet\,e^{\mathrm{i}\,\mathbf{k}_\bullet\cdot\boldsymbol{x}}\;(\mathfrak{s}\,\mathrm{i}\,\omega)\;\widehat{\mathbf{d}}\,, \tag{4.34b}$$

$$\boldsymbol{\tau}_\bullet^{\mathrm{pw}} \;=\; \mathrm{i}\,\omega\,\mathsf{s}_\bullet(\omega)\,e^{\mathrm{i}\mathbf{k}_\bullet\cdot\boldsymbol{x}}\left(2\,\mu_{\mathrm{fr}}\,\widehat{\mathbf{d}}\otimes\widehat{\mathbf{d}} + \Big(\underbrace{-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}}}_{\lambda_{\mathrm{fr}}} + M\,\alpha^2 \,+\, \upbeta_\bullet\,\alpha\,M\Big)\,\mathbf{Id}\right)\,, \tag{4.34c}$$

$$\mathrm{p}_\bullet^{\mathrm{pw}} \;=\; \mathrm{i}\,\omega\,\mathsf{s}_\bullet(\omega)\;(-M\,\upbeta_\bullet \;-\; M\,\alpha)\,e^{\mathrm{i}\mathbf{k}_\bullet\cdot\boldsymbol{x}}\,. \tag{4.34d}$$

with polarization given by

$$\begin{cases} \mathbf{k}_\bullet \;=\; \omega\,\mathsf{s}_\bullet(\omega)\,\widehat{\mathbf{d}} \quad , \quad \widehat{\mathbf{d}} = (\cos\alpha_{\mathrm{inc}}, \sin\alpha_{\mathrm{inc}}), \\ \mathsf{s}_\bullet(\omega) \text{ given by (4.9b) or (4.9c)}, \\ \beta_\bullet \;=\; -\dfrac{H\,\mathsf{s}_\bullet^2(\omega) - \rho_a}{\alpha\,M\,\mathsf{s}_\bullet^2(\omega) \,-\, \rho_{\mathrm{f}}} \qquad \textit{cf.}\,(4.26)\,. \end{cases} \tag{4.35}$$

We have the Jacobi-Anger expansion, see for e.g [26, eqn (2.17)],

$$e^{\mathrm{i}\,t\,\cos\varphi} = \sum_{k=-\infty}^{\infty} \mathrm{i}^k\,\mathrm{J}_k(t)\,e^{ik\varphi}. \tag{4.36}$$

The multipole expansion relative to the origin $0_{\mathbb{R}^2}$ is given by

$$\begin{aligned} e^{\mathrm{i}\,\kappa\boldsymbol{x}\,\cdot(\cos\alpha_{\mathrm{inc}},\sin\alpha_{\mathrm{inc}})} \;&=\; e^{\mathrm{i}\,\kappa\,r\,\cos(\theta-\alpha_{\mathrm{inc}})} \\ &=\; \sum_{k=-\infty}^{\infty} \mathrm{i}^k\,\mathrm{J}_k(\kappa\,r)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\,. \end{aligned} \tag{4.37}$$

Thus,

$$e^{\mathrm{i}\,\omega\,\mathsf{s}_\bullet\,\boldsymbol{x}\cdot\widehat{\mathbf{d}}} \;=\; \sum_{k=-\infty}^{\infty} \mathrm{i}^k\,\mathrm{J}_k(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\,. \tag{4.38}$$

We have

$$\begin{aligned} \mathbf{u}_\bullet^{\mathrm{pw}} \;&=\; \frac{\mathfrak{s}}{\mathsf{s}_\bullet}\nabla(e^{\mathrm{i}\,\mathbf{k}_\bullet\,\cdot\boldsymbol{x}})\,, \\ \Rightarrow \mathbf{u}_\bullet^{\mathrm{pw}} \;&=\; \frac{\mathfrak{s}}{\mathsf{s}_\bullet}\nabla\left(\sum_{k=-\infty}^{\infty} \mathrm{i}^k\,\mathrm{J}_k(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\right)\,. \end{aligned}$$

Next we use ∇ in polar coordinates $\nabla = \partial_r\mathbf{e}_r + \dfrac{1}{|\boldsymbol{x}|}\partial_\theta\mathbf{e}_\theta\,.$

$$\mathbf{u}_\bullet^{\mathrm{pw}} \;=\; \frac{\mathfrak{s}}{\mathsf{s}_\bullet}\sum_{k=-\infty}^{\infty} \mathrm{i}^k\,\omega\,\mathsf{s}_\bullet\,\mathrm{J}_k'(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\;\mathbf{e}_r \;+\; \frac{\mathfrak{s}}{\mathsf{s}_\bullet\;|\boldsymbol{x}|}\sum_{k=-\infty}^{\infty} \mathrm{i}^{k+1}\,k\,\mathrm{J}_k(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\;\mathbf{e}_\theta\,. \tag{4.39}$$

We obtain the same thing for $\mathbf{w}$,

$$\begin{aligned} \mathbf{w}_\bullet^{\mathrm{pw}} \;&=\; \beta_\bullet\,e^{\mathrm{i}\,\mathbf{k}_\bullet\cdot\boldsymbol{x}}\;(\mathfrak{s}\,\mathrm{i}\,\omega)\;\widehat{\mathbf{d}} \;=\; \frac{\beta_\bullet}{\mathsf{s}_\bullet}\nabla(e^{\mathrm{i}\,\mathfrak{s}\,\mathbf{k}_\bullet\,\cdot\boldsymbol{x}})\,, \\ \Rightarrow \mathbf{w}_\bullet^{\mathrm{pw}} \;&=\; \mathfrak{s}\,\frac{\beta_\bullet}{\mathsf{s}_\bullet}\nabla\left(\sum_{k=-\infty}^{\infty} \mathrm{i}^k\,\mathrm{J}_k(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\right)\,. \end{aligned}$$

In polar coordinates:

$$\mathbf{w}_\bullet^{\mathrm{pw}} \;=\; \mathfrak{s}\,\frac{\beta_\bullet}{\mathsf{s}_\bullet}\sum_{k=-\infty}^{\infty} \mathrm{i}^k\,\omega\,\mathsf{s}_\bullet\,\mathrm{J}_k'(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\,\mathbf{e}_r \;+\; \mathfrak{s}\,\frac{\beta_\bullet}{\mathsf{s}_\bullet\;|\boldsymbol{x}|}\sum_{k=-\infty}^{\infty} \mathrm{i}^{k+1}\,k\,\mathrm{J}_k(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\;\mathbf{e}_\theta\,. \tag{4.40}$$

For the stress tensor, equation (4.34), gives:

$$\boldsymbol{\tau}_\bullet^{\mathrm{pw}} \;=\; \frac{2\,\mu_{\mathrm{fr}}}{\mathrm{i}\,\omega\,\mathsf{s}_\bullet}\,\nabla^2 e^{\mathrm{i}\mathbf{k}_\bullet\cdot\boldsymbol{x}} \;+\; \Big(\underbrace{-\tfrac{2}{3}\mu_{\mathrm{fr}} \,+\, \mathrm{k}_{\mathrm{fr}}}_{\lambda_{\mathrm{fr}}} + M\,\alpha^2 \,+\, \beta_\bullet\,\alpha\,M\Big)\,\mathrm{i}\,\omega\,\mathsf{s}_\bullet\,e^{\mathrm{i}\mathbf{k}_\bullet\cdot\boldsymbol{x}}\,\mathbf{Id} \;\;:$$

We detail the components $\boldsymbol{\tau}_{rr}$ and $\boldsymbol{\tau}_{r\theta}$:

$$
\begin{aligned}
\boldsymbol{\tau}^{\mathrm{pw}}_{\bullet,rr} \;&=\; \frac{2\,\mu_{\mathrm{fr}}}{\mathrm{i}\,\omega\,\mathsf{s}_\bullet}\,\partial_{rr} e^{\mathrm{i}\mathbf{k}_\bullet\cdot\boldsymbol{x}} \;+\; \Big(\underbrace{-\tfrac{2}{3}\mu_{\mathrm{fr}} \,+\, \mathrm{k}_{\mathrm{fr}}}_{\lambda_{\mathrm{fr}}} + M\,\alpha^2 \,+\, \upbeta_\bullet\,\alpha\,M\Big)\,\mathrm{i}\,\omega\,\mathsf{s}_\bullet\,e^{\mathrm{i}\mathbf{k}_\bullet\cdot\boldsymbol{x}} \\
&=\; \frac{2\,\mu_{\mathrm{fr}}}{\mathrm{i}\,\omega\,\mathsf{s}_\bullet} \sum_{k=-\infty}^{\infty} \mathrm{i}^k\omega^2\,\mathsf{s}_\bullet^2\,\mathrm{J}_k''(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})} \\
&+\; \Big(\underbrace{-\tfrac{2}{3}\mu_{\mathrm{fr}} \,+\, \mathrm{k}_{\mathrm{fr}}}_{\lambda_{\mathrm{fr}}} + M\,\alpha^2 \,+\, \upbeta_\bullet\,\alpha\,M\Big)\,\mathrm{i}\,\omega\,\mathsf{s}_\bullet \sum_{k=-\infty}^{\infty} \mathrm{i}^k\,\mathrm{J}_k(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})} \\
&=\; 2\,\mu_{\mathrm{fr}} \sum_{k=-\infty}^{\infty} \mathrm{i}^{k-1}\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})} \,-\, \frac{2\,\mu_{\mathrm{fr}}}{\omega\,\mathsf{s}_\bullet} \sum_{k=-\infty}^{\infty} \mathrm{i}^{k-1}\,\mathrm{J}_k(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})} \\
&-\; 2\,\mu_{\mathrm{fr}} \sum_{k=-\infty}^{\infty} \mathrm{i}^{k-1}\omega\,\mathsf{s}_\bullet\,\mathrm{J}_k(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})} \,+\, \frac{2\,\mu_{\mathrm{fr}}}{\omega\,\mathsf{s}_\bullet} \sum_{k=-\infty}^{\infty} \mathrm{i}^{k-1}k^2\,\mathrm{J}_k(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})} \\
&+\; \Big(\underbrace{-\tfrac{2}{3}\mu_{\mathrm{fr}} \,+\, \mathrm{k}_{\mathrm{fr}}}_{\lambda_{\mathrm{fr}}} + M\,\alpha^2 \,+\, \upbeta_\bullet\,\alpha\,M\Big)\,\mathrm{i}\,\omega\,\mathsf{s}_\bullet \sum_{k=-\infty}^{\infty} \mathrm{i}^k\,\mathrm{J}_k(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\,,
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\tau}^{\mathrm{pw}}_{\bullet,r\theta} \;&=\; \frac{2\,\mu_{\mathrm{fr}}}{\mathrm{i}\,\omega\,\mathsf{s}_\bullet}\left(\frac{\partial_{\theta r}}{|\boldsymbol{x}|}e^{\mathrm{i}\mathbf{k}_\bullet\cdot\boldsymbol{x}} \,-\, \frac{\partial_\theta}{|\boldsymbol{x}|^2}e^{\mathrm{i}\mathbf{k}_\bullet\cdot\boldsymbol{x}}\right) \\
&=\; \frac{2\,\mu_{\mathrm{fr}}}{\mathrm{i}\,\omega\,\mathsf{s}_\bullet}\left(\frac{1}{|\boldsymbol{x}|}\sum_{k=-\infty}^{\infty}\omega\,\mathsf{s}_\bullet\,\mathrm{i}^{k+1}\,k\,\mathrm{J}_k'(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})} \,-\, \frac{1}{|\boldsymbol{x}|^2}\sum_{k=-\infty}^{\infty}\mathrm{i}^{k+1}\,k\,\mathrm{J}_k(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\right) \\
&=\; \sum_{k=-\infty}^{\infty}\frac{2\,\mu_{\mathrm{fr}}\,\mathrm{i}^k\,k}{|\boldsymbol{x}|}\,\mathrm{J}_k'(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})} \,-\, \sum_{k=-\infty}^{\infty}\frac{2\,\mu_{\mathrm{fr}}\mathrm{i}^k\,k}{\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\,.
\end{aligned}
$$

For the pressure, we obtain directly

$$
\mathrm{p}^{\mathrm{pw}}_\bullet \;=\; \mathrm{i}\,\omega\,\mathsf{s}_\bullet\,\left(-M\,\upbeta_\bullet \;-\; M\,\alpha\right)\sum_{k=-\infty}^{\infty}\mathrm{i}^k\,\mathrm{J}_k(\omega\,\mathsf{s}_\bullet\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\,.
$$

For a transverse wave The admissible transverse planewave allowed in an isotropic poroelastic medium from (4.30) is expressed as:

$$
\begin{aligned}
\mathbf{u}^{\mathrm{pw}}_{\mathrm{S}} \;&=\; e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}}\;(\mathfrak{s}\,\mathrm{i}\,\omega)\;\widehat{\mathbf{d}} \;=\; -\,\frac{\mathfrak{s}}{\mathsf{s}_{\mathrm{S}}}\mathrm{curl}\left(e^{\mathrm{i}\,\mathfrak{s}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}}\right)\,, \\
\mathbf{w}^{\mathrm{pw}}_{\mathrm{S}} \;&=\; \upbeta_{\mathrm{S}}\,e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}}\;(\mathfrak{s}\,\mathrm{i}\,\omega)\;\widehat{\mathbf{d}} \;=\; -\,\mathfrak{s}\,\frac{\upbeta_{\mathrm{S}}}{\mathsf{s}_{\mathrm{S}}}\mathrm{curl}\left(e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}}\right)\,, \\
\boldsymbol{\tau}^{\mathrm{pw}}_{\mathrm{S}} \;&=\; \mathrm{i}\,\omega\,\mathsf{s}_{\mathrm{S}}\,e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}}\,\mu_{\mathrm{fr}}\left(\widehat{\mathbf{k}}\otimes\widehat{\mathbf{d}} \,+\, \widehat{\mathbf{d}}\otimes\widehat{\mathbf{k}}\right), \\
\mathrm{p}^{\mathrm{pw}}_{\mathrm{S}} &= 0\,,
\end{aligned}
\tag{4.41}
$$

with the polarization given by

$$
\begin{cases}
\mathbf{k}_{\mathrm{S}} \;=\; \omega\,\mathsf{s}_{\mathrm{S}}(\omega)\,\widehat{\mathbf{k}} \quad , \\[1ex]
\widehat{\mathbf{k}} = (\cos\alpha_{\mathrm{inc}},\sin\alpha_{\mathrm{inc}}) \quad \widehat{\mathbf{d}} = (-\sin\alpha_{\mathrm{inc}},\cos\alpha_{\mathrm{inc}}), \\[1ex]
\mathsf{s}_{\mathrm{S}}(\omega) \text{ given by (4.9a)}\,, \\[1ex]
\upbeta_{\mathrm{S}} \;=\; \dfrac{\rho_{\mathrm{f}}}{\rho_{\mathrm{dyn}}(\omega)} \quad \textit{cf.}\,(4.19)\,.
\end{cases}
\tag{4.42}
$$

Recall that the multipole expansion relative to the origin $0_{\mathbb{R}^2}$ is given by

$$
e^{\mathrm{i}\,\kappa\boldsymbol{x}\cdot(\cos\alpha_{\mathrm{inc}},\sin\alpha_{\mathrm{inc}})} \;=\; \sum_{k=-\infty}^{\infty}\mathrm{i}^k\,\mathrm{J}_k(\kappa\,r)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\,.
\tag{4.43}
$$

We have

$$\begin{aligned}
\mathbf{u}_{\mathrm{S}}^{\mathrm{pw}} &= -\frac{\mathfrak{s}}{\mathsf{s}_{\mathrm{S}}}\mathbf{curl}(e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}}),\\
\Rightarrow \mathbf{u}_{\mathrm{S}}^{\mathrm{pw}} &= -\frac{\mathfrak{s}}{\mathsf{s}_{\mathrm{S}}}\mathbf{curl}\left(\sum_{k=-\infty}^{\infty}\mathrm{i}^k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\right).
\end{aligned}$$

Next we use **curl** in polar coordinates $\mathbf{curl} = \dfrac{1}{r}\partial_\theta \mathbf{e}_r - -\partial_r \mathbf{e}_\theta$

$$\begin{aligned}
\mathbf{u}_{\mathrm{S}}^{\mathrm{pw}} &= -\frac{\mathfrak{s}}{\mathsf{s}_{\mathrm{S}}\;|\boldsymbol{x}|}\sum_{k=-\infty}^{\infty}\mathrm{i}^{k+1}\,k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\;\mathbf{e}_r\\
&\quad+\frac{\mathfrak{s}}{\mathsf{s}_{\mathrm{S}}}\sum_{k=-\infty}^{\infty}\mathrm{i}^k\,\omega\,\mathsf{s}_{\mathrm{S}}\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\;\mathbf{e}_\theta\,.
\end{aligned}$$

Equivalently for $\mathbf{w}$,

$$\begin{aligned}
\mathbf{w}_{\mathrm{S}}^{\mathrm{pw}} &= \beta_{\mathrm{S}}\,e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}}\;(\mathfrak{s}\,\mathrm{i}\,\omega)\;\widehat{\mathbf{d}} \;=\; -\,\mathfrak{s}\,\frac{\beta_{\mathrm{S}}}{\mathsf{s}_{\mathrm{S}}}\mathrm{curl}\left(e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}}\right),\\
\Rightarrow \mathbf{w}_{\mathrm{S}}^{\mathrm{pw}} &= -\,\mathfrak{s}\,\frac{\beta_{\mathrm{S}}}{\mathsf{s}_{\mathrm{S}}}\mathbf{curl}\left(\sum_{k=-\infty}^{\infty}\mathrm{i}^k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\right).\\
\mathbf{w}_{\mathrm{S},r}^{\mathrm{pw}} &= -\,\mathfrak{s}\,\frac{\beta_{\mathrm{S}}}{\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|}\sum_{k=-\infty}^{\infty}\mathrm{i}^{k+1}k\;\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\,. \qquad (4.44)
\end{aligned}$$

For $\boldsymbol{\tau}$, we have from equation (4.41)

$$\begin{aligned}
\boldsymbol{\tau}_{\mathrm{S}}^{\mathrm{pw}} &= \mathrm{i}\,\omega\,\mathsf{s}_{\mathrm{S}}\,e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}}\,\mu_{\mathrm{fr}}\left(\widehat{\mathbf{k}}\otimes\widehat{\mathbf{d}}+\widehat{\mathbf{d}}\otimes\widehat{\mathbf{k}}\right)\\
&= \mathrm{i}\,\omega\,\mathsf{s}_{\mathrm{S}}\,e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}}\,\mu_{\mathrm{fr}}\begin{pmatrix}-2\cos\alpha_{\mathrm{inc}}\sin\alpha_{\mathrm{inc}} & \cos\alpha_{\mathrm{inc}}^2-\sin\alpha_{\mathrm{inc}}^2\\ \cos\alpha_{\mathrm{inc}}^2-\sin\alpha_{\mathrm{inc}}^2 & 2\cos\alpha_{\mathrm{inc}}\sin\alpha_{\mathrm{inc}}\end{pmatrix}.
\end{aligned}$$

The components $\boldsymbol{\tau}_{rr}$ and $\boldsymbol{\tau}_{r\theta}$ are:

$$\begin{aligned}
\boldsymbol{\tau}_{\mathrm{S},rr}^{\mathrm{pw}} &= -2\frac{\mu_{\mathrm{fr}}}{\mathrm{i}\omega\mathsf{s}_{\mathrm{S}}}\,\frac{\partial_{r\theta}(e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}})}{|\boldsymbol{x}|}\\
&= \sum_{k=-\infty}^{\infty}-2\frac{\mu_{\mathrm{fr}}}{\mathrm{i}\omega\mathsf{s}_{\mathrm{S}}|\boldsymbol{x}|}\left(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathrm{i}^{k+1}\,k\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\right)\\
&= \sum_{k=-\infty}^{\infty}-2\frac{\mu_{\mathrm{fr}}}{|\boldsymbol{x}|}\,\mathrm{i}^k\,k\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})},
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\tau}_{\mathrm{S},r\theta}^{\mathrm{pw}} &= -\frac{\mu_{\mathrm{fr}}}{\mathrm{i}\omega\mathsf{s}_{\mathrm{S}}}\left(\frac{\partial_{\theta\theta}(e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}})}{|\boldsymbol{x}|^2}+\frac{\partial_r(e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}})}{|\boldsymbol{x}|}-\partial_{rr}(e^{\mathrm{i}\,\mathbf{k}_{\mathrm{S}}\cdot\boldsymbol{x}})\right)\\
&= \sum_{k=-\infty}^{\infty}\frac{\mu_{\mathrm{fr}}}{\mathrm{i}\omega\mathsf{s}_{\mathrm{S}}}\left(\frac{\mathrm{i}^k\,k^2}{|\boldsymbol{x}|^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}-\frac{\omega\,\mathsf{s}_{\mathrm{S}}\,\mathrm{i}^k}{|\boldsymbol{x}|}\left(\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\right)\right.\\
&\quad\left.+\omega^2\,\mathsf{s}_{\mathrm{S}}^2\,\mathrm{i}^k\,\mathrm{J}_k''(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\right)\\
&= \sum_{k=-\infty}^{\infty}\frac{\mu_{\mathrm{fr}}\,\mathrm{i}^{k-1}\,k^2}{\omega\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}-\sum_{k=-\infty}^{\infty}\frac{\mu_{\mathrm{fr}}\,\mathrm{i}^{k-1}}{|\boldsymbol{x}|}\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\\
&\quad+\sum_{k=-\infty}^{\infty}\mu_{\mathrm{fr}}\,\mathrm{i}^{k-1}\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}-\sum_{k=-\infty}^{\infty}\frac{\mu_{\mathrm{fr}}\,k}{\omega\,\mathsf{s}_{\mathrm{S}}}\,\mathrm{i}^{k-1}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\\
&\quad-\sum_{k=-\infty}^{\infty}\mu_{\mathrm{fr}}\,\omega\,\mathsf{s}_{\mathrm{S}}\,\mathrm{i}^{k-1}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}+\sum_{k=-\infty}^{\infty}\frac{\mu_{\mathrm{fr}}\,k^2}{\omega^2\,\mathsf{s}_{\mathrm{S}}^2}\,\mathrm{i}^{k-1}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\,.
\end{aligned}$$

5 Properties of slowness

In this section, we describe the properties of the slowness in a poroelastic medium. These properties are dependent on the convention we use for time-derivative in the frequency domain. Moreover, the value of the viscosity in the domain has an influence on these properties. We distinct two cases for the viscosity: one with vanishing viscosity, using the assumptions of section 3.1.4, and one with a positive viscosity. In a first time, we study the properties of $\mathsf{s}_\bullet^2$, then we propose a definition of the slownesses.

5.1 Properties of slowness square

In section 4.2, we presented the expressions of the slownesses for the three waves, which depend on matrices A, B, C, that we defined. The values of thoses matrices hence give properties of the slowness. In the case with no viscosity, the matrices are real. Here we present the properties of the slowness square from equation (4.9).

5.1.1 Zero viscosity

From the discussion in subsection 3.1.4 and Remark 2, the dynamic density $\rho_{\text{dyn}}(\omega)$ at vanishing viscosity takes the form

$$\lim_{\substack{\eta\to 0,\\ \text{fixed }\omega>0}} \rho_{\text{dyn}} \;=\; \rho_{\text{dyn}}^{\text{VV}} \;=\; \frac{\rho_{\text{f}}}{\phi}\,\alpha_\infty$$

cf. (3.22) and for zero-viscosity at low-frequency, cf. (B.7), it is replaced by

$$\lim_{\substack{\eta\to 0,\\ \text{fixed }\omega>0}} \rho_{\text{dyn}}^{\text{LF}} \;=\; \rho_{\text{VV}}^{\text{LF}} \;=\; \rho_{\mathsf{w}} \;=\; \frac{\rho_{\text{f}}}{\phi}\,\alpha_\infty\,(1+\tfrac{2}{m}) \;>\; 0\,.$$

Denote by t the ‘formal’ tortuosity defined as

$$\begin{aligned} &\mathsf{t}=\alpha_\infty && \text{for zero-viscosity,}\\ \text{or}\qquad &\mathsf{t}=\alpha_\infty(1+\tfrac{2}{m}) && \text{for formal zero-viscosity in low-frequency}\,. \end{aligned} \tag{5.1}$$

Using this notation, for both cases, the dynamic $\rho_{\text{dyn}}(\omega)$ for vanishing viscosity takes the form

$$\rho_{\text{VV}} \;:=\; \frac{\rho_{\text{f}}}{\phi}\mathsf{t}\,. \tag{5.2}$$

By its definition (2.1), the porosity ϕ satisfies

$$0\;\leq\phi\;\leq\;1\,. \tag{5.3}$$

As a physical quantity, we can also assume that

$$\mathsf{t}\;\geq\;1\,. \tag{5.4}$$

Note that the second inequality is automatic for the case of zero-viscosity at low-frequency (with the natural assumption in both cases that $\alpha_\infty\geq 0$). We also assume that t and ϕ are not 1 at the same time, i.e.

$$(1-\mathsf{t})^2\;+\;(1-\phi)^2\;>\;0\,. \tag{5.5}$$

At zero viscosity (for either low or regular regime), we use ρ^{VV} introduced in (5.2) in the definition of matrices A and C in (4.10) and (4.11). Theses matrices appear in the definition of the slowness square (4.9). In this case, all of the quantities are real.

Proposition 4. *Under hypothesis* (5.3)– (5.5)*, we have*

1. *Matrix*

$$A_0\;:=\;\begin{pmatrix}\rho_a & \rho_{\text{f}}\\ \rho_{\text{f}} & \rho_{\text{VV}}\end{pmatrix}$$

is symmetric and positive definite.

2. *The matrix*

$$C_0\;:=\;B^{cof}A_0$$

is diagonalizable with two positive eigenvalues.

Proof. For the following proof, we first recall from (4.10),

$$B^{\mathrm{cof}} \;=\; \begin{pmatrix} M & -\alpha\, M \\ -\alpha M & H \end{pmatrix}.$$

Statement 1 The proof follows [17, Rmk 5.2.1]. We substitute in A_0 the definition of ρ_a given in (2.2) and that of vanishing viscosity density ρ_{VV} in (5.2),

$$\begin{aligned} \det A_0 \;&=\; \rho_a\,\rho_{\mathrm{VV}} \;-\; \rho_{\mathrm{f}}^2 \\ &=\; ((1-\phi)\,\rho_{\mathrm{s}} \;+\; \phi\,\rho_{\mathrm{f}})\,\frac{\rho_{\mathrm{f}}}{\phi}\,\mathfrak{t} \;-\; \rho_{\mathrm{f}}^2 \\ &=\; \big(\mathfrak{t} \;-\; 1\big)\,\rho_{\mathrm{f}}^2 \;+\; \frac{(1-\phi)}{\phi}\,\mathfrak{t}\,\rho_{\mathrm{s}}\,\rho_{\mathrm{f}}\,. \end{aligned} \tag{5.6}$$

Under hypothesis (5.3)– (5.5), we have

$$\det A_0 \;>\; 0\,.$$

At the same time we have

$$\operatorname{tr} A_0 \;>\; 0\,,$$

since A_0 is symmetric, and thus diagonalizable. With its determinant and trace positive, its two eigenvalues are positive.

Statement 2 The matrix B is always real, symmetric. Due to (2.7)

$$M \;>\; 0 \quad,\quad H \;=\; \lambda_{\mathrm{fr}} + 2\mu_{\mathrm{fr}} + \alpha^2 M \;>\; 0\,,$$

we have

$$\begin{aligned} \det B \;&=\; M(\lambda_{\mathrm{fr}} + 2\mu_{\mathrm{fr}}) \;>\; 0 \\ \operatorname{tr} B \;&=\; M \;+\; H \;\;>\; 0\,. \end{aligned}$$

Since B is symmetric, this means that B is diagonalizable with its eigenvalues positive (and thus positive definite). We can define its square root denoted by $\tilde{B}$,

$$\tilde{B}^2 \;=\; B\,.$$

Note that $\tilde{B}$ is also symmetric and positive definite (since $B = QDQ^t$ with an orthogonal matrix Q i.e. $Q^{-1} = Q^t$, then $\tilde{B} = Q\sqrt{D}Q^t$ and $\tilde{B}^t = \tilde{B}$), and so is its inverse $(\tilde{B})^{-1}$.

We next show that $B^{-1}A_0$ is similar to the symmetric matrix $\tilde{B}^{-1}A_0\tilde{B}^{-1}$. We first have

$$\tilde{B}\left(B^{-1}\,A_0\right)\tilde{B}^{-1} \;=\; \tilde{B}^{-1}\,A_0\,\tilde{B}^{-1}\,.$$

The latter matrix satisfies,

$$\left(\tilde{B}^{-1}\,A_0\,\tilde{B}^{-1}\right)^t \;=\; \left(\tilde{B}^{-1}\right)^t A_0^t \left(\tilde{B}^{-1}\right)^t,$$

and is thus symmetric since A_0 is symmetric and $\tilde{B}^{-1}$ is symmetric. This means that $\tilde{B}^{-1}A_0\tilde{B}^{-1}$ is diagonalizable. Furthermore, its eigenvalues are positive. This is seen by showing that $\tilde{B}^{-1}A_0\tilde{B}^{-1}$ is positive definite,

$$\left\langle\, \tilde{B}^{-1}\,A_0\,\tilde{B}^{-1}x\;,\;x\,\right\rangle \;=\; \left\langle A_0\,\tilde{B}^{-1}\,x\;,\;(\tilde{B}^{-1})^t\,x\right\rangle \;=\; \left\langle A_0\,\tilde{B}^{-1}\,x\;,\;(\tilde{B}^{-1})\,x\right\rangle \;>\; 0\,.$$

The last inequality is due to the definite positivity of A_0 discussed in statement 1. By similarity, these properties are transferred to matrix $B^{-1}A_0$, and thus C_0. □

Remark 5. When $\eta = 0$, all of the concerned quantities are real. A and B are symmetric, hence diagonalizable. The material in Table 2 with zero viscosity (Sandstone and Shale) satisfies the condition guaranteeing the positive definiteness of matrices A and B. For A, this means

$$\rho_a + \rho_{\mathrm{dyn}} > 0 \quad,\quad \rho_a\rho_{\mathrm{dyn}} - \rho_{\mathrm{f}}^2 > 0\,.$$

For B, this means

$$H + M > 0 \quad,\quad HM > \alpha^2 M^2\,.$$

△

5.1.2 With viscosity

Proposition 5. *1. We assume*

$$
\begin{aligned}
&a) \qquad \rho_a \operatorname{Re}\rho_{\text{dyn}} \;-\; \rho_{\text{f}}^2 \;>\; 0\,, && (5.7a)\\
&b) \qquad \rho_a - 2\alpha\rho_{\text{f}} \;>\; 0\,, && (5.7b)\\
&c) \qquad \rho_a - 2\alpha\rho_{\text{f}} + \frac{H}{M}\operatorname{Re}\rho_{\text{dyn}} \;>\; 2\frac{\det B}{HM}\,\rho_a\,. && (5.7c)
\end{aligned}
$$

Under the above hypotheses, the following slownesses square, as defined in (4.9), satisfy

$$\operatorname{Re}\mathsf{s}_{\text{S}}^2 > 0 \quad , \quad -\mathfrak{s}\operatorname{Im}\mathsf{s}_{\text{S}}^2 \;>\; 0\,, \tag{5.8}$$

while

$$\operatorname{Re}\mathsf{s}_{\text{B}}^2 > 0 \quad , \quad -\mathfrak{s}\operatorname{Im}\mathsf{s}_{\text{B}}^2 > 0\,, \tag{5.9}$$

and that

$$|\mathsf{s}_{\text{B}}^2| > |\mathsf{s}_{\text{P}}^2|\,. \tag{5.10}$$

2. To make further statement on the P*-slowness, in addition to (5.7), we assume that*

$$
\begin{aligned}
&d) \qquad \rho_a \;-\; 2\alpha\,\rho_{\text{f}} \;+\; \frac{H}{M}\operatorname{Re}\rho_{\text{dyn}} \;<\; \frac{1}{4}\,\frac{H}{M}\,|\operatorname{Im}\rho_{\text{dyn}}|\,. && \textit{Assumption 4}\,, && (5.11a)\\
&e) \qquad \rho_a - 2\alpha\rho_{\text{f}} + \frac{H}{M}\operatorname{Re}\rho_{\text{dyn}} \;<\; \frac{4}{2-\sqrt{3}}\,\frac{\det B}{H\,M}\,\rho_a && \textit{Assumption 5}\,. && (5.11b)
\end{aligned}
$$

Then, we have

$$\operatorname{Re}\mathsf{s}_{\text{P}}^2 > 0 \quad , \quad -\mathfrak{s}\operatorname{Im}\mathsf{s}_{\text{P}}^2 > 0\,. \tag{5.12}$$

Proof. We recall from (3.19) that

$$\operatorname{Re}\rho_{\text{dyn}} > 0 \quad , \quad \operatorname{Im}\rho_{\text{dyn}} \begin{cases} > 0 & \text{in Convention 1} \\ < 0 & \text{in Convention 2}\end{cases} \,. \tag{5.13}$$

Properties of S **slowness square** We consider the expression

$$\mu_{\text{fr}}\;\mathsf{s}_{\text{S}}^2 \;=\; \frac{\det A}{\rho_{\text{dyn}}} \;=\; \frac{\rho_a\,|\rho_{\text{dyn}}|^2 \;-\; \rho_{\text{f}}^2\,\overline{\rho_{\text{dyn}}}}{|\rho_{\text{dyn}}|^2} \;=\; \rho_a \;-\; \rho_{\text{f}}^2\,\frac{\operatorname{Re}\rho_{\text{dyn}}}{|\rho_{\text{dyn}}|^2} \;+\; \mathrm{i}\,\rho_{\text{f}}^2\,\frac{\operatorname{Im}\rho_{\text{dyn}}}{|\rho_{\text{dyn}}|^2}\,.$$

Since

$$
\begin{aligned}
(\operatorname{Re}\rho_{\text{dyn}})^2 \;<\; |\rho_{\text{dyn}}|^2 \quad &\Rightarrow \quad \operatorname{Re}\rho_{\text{dyn}} \;<\; \frac{|\rho_{\text{dyn}}|^2}{\operatorname{Re}\rho_{\text{dyn}}}\,,\\
&\Rightarrow \rho_a\operatorname{Re}\rho_{\text{dyn}} \;-\; \rho_{\text{f}}^2 \;<\; \rho_a\frac{|\rho_{\text{dyn}}|^2}{\operatorname{Re}\rho_{\text{dyn}}} \;-\; \rho_{\text{f}}^2\,.
\end{aligned}
$$

Together with assumption (5.7a) that gives

$$\rho_a\;\operatorname{Re}\rho_{\text{dyn}} - \rho_{\text{f}}^2 \quad > 0\,,$$

we obtain

$$0 \;<\; \rho_a\;\operatorname{Re}\rho_{\text{dyn}} \;-\; \rho_{\text{f}}^2 \;<\; \rho_a\frac{|\rho_{\text{dyn}}|^2}{\operatorname{Re}\rho_{\text{dyn}}} \;-\; \rho_{\text{f}}^2 \qquad \text{and} \qquad \operatorname{Re}\mathsf{s}_{\text{S}}^2 \;>\; 0 \quad , \quad (-\mathfrak{s})\operatorname{Im}\mathsf{s}_{\text{S}}^2 \;>\; 0\,.$$

Properties of B **slowness square** We consider the quantities

$$\operatorname{tr}C \;=\; M\rho_a - 2\alpha M\rho_{\text{f}} + H\rho_{\text{dyn}} \;=\; M(\rho_a - 2\alpha\rho_{\text{f}}) \;+\; H\,\operatorname{Re}\rho_{\text{dyn}} \;+\; \mathrm{i}\,H\,\operatorname{Im}\rho_{\text{dyn}}\,.$$

Thus with assumption (5.7b) and (5.13), we have

$$\operatorname{Re}\operatorname{tr}C > 0 \quad , \quad -\mathfrak{s}\operatorname{Im}\operatorname{tr}C \;>\; 0\,. \tag{5.14}$$

Next we consider

$$(\operatorname{tr} C)^2 - 4(\det A)(\det B) \;=\; (M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\,\operatorname{Re}\rho_{\mathrm{dyn}})^2 - 4(\rho_a\,\operatorname{Re}\rho_{\mathrm{dyn}} - \rho_{\mathrm{f}}^2)\det B \;-\; H^2\,(\operatorname{Im}\rho_{\mathrm{dyn}})^2$$
$$+\;2\,\mathrm{i}\,H\,(M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\,\operatorname{Re}\rho_{\mathrm{dyn}})\operatorname{Im}\rho_{\mathrm{dyn}} - 4\mathrm{i}(\det B)\,\rho_a\operatorname{Im}\rho_{\mathrm{dyn}}\,. \tag{5.15}$$

With assumption (5.7c),

$$(\operatorname{Im}\rho_{\mathrm{dyn}})\;\operatorname{Im}\big((\operatorname{tr} C)^2 - 4(\det A)(\det B)\big) \;>\; 0\,.$$

Using the square root given by $\sqrt{\bullet}$, we then have

$$\operatorname{Re}\sqrt{(\operatorname{tr} C)^2 - 4(\det A)(\det B)} > 0 \quad , \quad (\operatorname{Im}\rho_{\mathrm{dyn}})\;\operatorname{Im}\sqrt{(\operatorname{tr} C)^2 - 4(\det A)(\det B)} \;>\; 0\,, \tag{5.16}$$

which is equivalent to

$$\operatorname{Re}\sqrt{(\operatorname{tr} C)^2 - 4(\det A)(\det B)} > 0 \quad , \quad -\mathfrak{s}\;\operatorname{Im}\sqrt{(\operatorname{tr} C)^2 - 4(\det A)(\det B)} \;>\; 0\,. \tag{5.17}$$

We next compare the magnitude of square B-slowness and P-slowness. As a result of (5.14) and (5.17), with $a, b, \tilde{a}, \tilde{b} > 0$, we can write

$$\operatorname{tr} C \;=\; a \;-\; \mathrm{i}\mathfrak{s}\,b \quad , \quad (\operatorname{tr} C)^2 - 4(\det A)(\det B) \;=\; \tilde{a} \;-\; \mathrm{i}\mathfrak{s}\,\tilde{b}\,. \tag{5.18}$$

With $a, b, \tilde{a}, \tilde{b} > 0$, we consider

$$z_\pm \;:=\; a \;-\; \mathrm{i}\,\mathfrak{s}\,b \;\pm\; (\tilde{a} \;-\; \mathrm{i}\,\mathfrak{s}\,\tilde{b}) \;=\; a \pm \tilde{a} \;-\; \mathrm{i}\mathfrak{s}\,(b \pm \tilde{b})\,.$$

We have

$$|z_\pm|^2 \;=\; (a \pm \tilde{a})^2 \;+\; (b \pm \tilde{b})^2\,.$$

Since

$$|a - \tilde{a}| \;<\; a + \tilde{a} \quad , \quad |b - \tilde{b}| < b + \tilde{b}\,,$$

we obtain

$$|z_+| \;>\; |z_-|\,.$$

For this reason, as defined in (4.9), we have

$$|\mathsf{s}_{\mathrm{P}}^2| \;<\; |\mathsf{s}_{\mathrm{B}}^2|\,.$$

In addition,

$$\operatorname{Re}\mathsf{s}_{\mathrm{B}}^2 \;>\; 0 \quad , \quad -\mathfrak{s}\;\operatorname{Re}\mathsf{s}_{\mathrm{B}}^2 \;>\; 0\,.$$

Properties of P **slowness square** Due to the assumption (5.7a), which gives $\rho_a\,\operatorname{Re}\rho_{\mathrm{dyn}} \;-\; \rho_{\mathrm{f}}^2 \;>\; 0$, we have

$$(M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\,\operatorname{Re}\rho_{\mathrm{dyn}})^2 - 4(\rho_a\,\operatorname{Re}\rho_{\mathrm{dyn}} - \rho_{\mathrm{f}}^2)\det B \;<\; (M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\,\operatorname{Re}\rho_{\mathrm{dyn}})^2\,.$$

By assumption (5.11a),

$$(M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\,\operatorname{Re}\rho_{\mathrm{dyn}})^2 < \frac{1}{4}\,H^2(\operatorname{Im}\rho_{\mathrm{dyn}})^2,$$

we thus have

$$(M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\,\operatorname{Re}\rho_{\mathrm{dyn}})^2 - 4(\rho_a\,\operatorname{Re}\rho_{\mathrm{dyn}} - \rho_{\mathrm{f}}^2)\det B < (M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\,\operatorname{Re}\rho_{\mathrm{dyn}})^2 < \frac{1}{4}\,H^2(\operatorname{Im}\rho_{\mathrm{dyn}})^2\,.$$

The quantity defined as

$$\epsilon \;:=\; \frac{(M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\,\operatorname{Re}\rho_{\mathrm{dyn}})^2 - 4(\rho_a\,\operatorname{Re}\rho_{\mathrm{dyn}} - \rho_{\mathrm{f}}^2)\det B}{H^2(\operatorname{Im}\rho_{\mathrm{dyn}})^2}\,, \tag{5.19}$$

then satisfies

$$0 \;<\; \epsilon \;<\; \frac{1}{4}\,. \tag{5.20}$$

For this reason, from the full expression in (5.15), we can write

$$\operatorname{Re}\big((\operatorname{tr} C)^2 \;-\; 4(\det A)(\det B)\big) \;=\; H^2\,(\operatorname{Im}\rho_{\mathrm{dyn}})^2\,(\epsilon - 1) \;<\; 0\,. \tag{5.21}$$

Similarly, we have

$$M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\ \mathrm{Re}\,\rho_{\mathrm{dyn}} \ <\ \frac{1}{4}H|\mathrm{Im}\,\rho_{\mathrm{dyn}}|\,,$$
$$\Rightarrow\quad 2H|\mathrm{Im}\,\rho_{\mathrm{dyn}}|\,(M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\ \mathrm{Re}\,\rho_{\mathrm{dyn}}) < \frac{1}{2}H^2|\mathrm{Im}\,\rho_{\mathrm{dyn}}|^2\,.$$

As a result, using equation (5.15), we can write

$$\begin{aligned}&2\ H\,\mathrm{Im}\,\rho_{\mathrm{dyn}}\,(M\rho_a - 2\alpha M\rho_{\mathrm{f}}\ +\ H\ \mathrm{Re}\,\rho_{\mathrm{dyn}})\ -\ 4(\det B)\,\rho_a\,\mathrm{Im}\,\rho_{\mathrm{dyn}}\\ &\qquad <\ 2\ H\,\mathrm{Im}\,\rho_{\mathrm{dyn}}\,(M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\ \mathrm{Re}\,\rho_{\mathrm{dyn}})\ <\ \frac{1}{2}\,H^2\,|\mathrm{Im}\,\rho_{\mathrm{dyn}}|^2\,.\end{aligned}$$

Now, for the quantity

$$\tilde{\epsilon}\ :=\ \frac{2\ H\,(M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\ \mathrm{Re}\,\rho_{\mathrm{dyn}})\ -\ 4(\det B)\,\rho_a}{H^2\,|\mathrm{Im}\,\rho_{\mathrm{dyn}}|}\,, \tag{5.22}$$

then

$$0\ <\ \tilde{\epsilon}\ <\ \frac{1}{2}\,. \tag{5.23}$$

we can write

$$\mathrm{Im}((\mathrm{tr}\,C)^2\ -\ 4(\det A)(\det B))\ =\ -\mathfrak{s}\,H^2\,(\mathrm{Im}\,\rho_{\mathrm{dyn}})^2\tilde{\epsilon}\,. \tag{5.24}$$

Denote by

$$\hat{\epsilon}\ :=\ \frac{\tilde{\epsilon}}{1-\epsilon}\,. \tag{5.25}$$

Since

$$\tilde{\epsilon} + \epsilon < 1 \Leftrightarrow \tilde{\epsilon} < 1 - \epsilon\,,$$

the first statement is true due to (5.20) and (5.23). We thus have

$$0\ <\ \hat{\epsilon}\ <\ 1\,. \tag{5.26}$$

In fact, from (5.20),

$$\tfrac{3}{4}\ <\ 1-\epsilon\ <\ 1\quad\Rightarrow\quad \hat{\epsilon} < \frac{4}{3}\tilde{\epsilon}\,. \tag{5.27}$$

Putting together (5.21) and (5.24), we write

$$\begin{aligned}\sqrt{(\mathrm{tr}\,C)^2\ -\ 4(\det A)(\det B)}\ &=\ H\,(-\mathfrak{s}\ \mathrm{Im}\,\rho_{\mathrm{dyn}})\ \sqrt{1-\epsilon}\ \sqrt{-1\ -\ \mathrm{i}\,\mathfrak{s}\,\frac{\tilde{\epsilon}}{1-\epsilon}}\\ &\overset{\text{Remark 6}}{=}\ -\mathfrak{s}\,H\,(\mathrm{Im}\,\rho_{\mathrm{dyn}})\ \sqrt{1-\epsilon}\ (-\mathfrak{s})\ \mathrm{i}\sqrt{1\ +\ \mathrm{i}\mathfrak{s}\,\hat{\epsilon}}\\ &=\ H\,(\mathrm{Im}\,\rho_{\mathrm{dyn}})\ \sqrt{1-\epsilon}\ \mathrm{i}\,\Big(1\ +\ \tfrac{1}{2}\mathrm{i}\,\mathfrak{s}\,\hat{\epsilon}\ +\ \mathsf{O}(\hat{\epsilon}^2)\Big)\\ &=\ \tfrac{1}{2}H|\mathrm{Im}\,\rho_{\mathrm{dyn}}|\,\tilde{\epsilon}^{1/2}\hat{\epsilon}^{1/2}\ \ +\ \mathrm{i}\,H\,(\mathrm{Im}\,\rho_{\mathrm{dyn}})\,\sqrt{1-\epsilon}\ +\ \mathsf{O}(\hat{\epsilon}^{3/2}\,\tilde{\epsilon}^{1/2})\,.\end{aligned}$$

In the first and last equality, we have used $|\mathrm{Im}\,\rho_{\mathrm{dyn}}| = -\mathfrak{s}\,\mathrm{Im}\,\rho_{\mathrm{dyn}}$. Thus

$$\begin{aligned}(\det B)\,\mathsf{s}_{\mathrm{P}}^2\ &=\ \mathrm{tr}\,C\ -\ 4\sqrt{(\mathrm{tr}\,C)^2\ -\ 4(\det A)(\det B)}\\ &=\ (M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\ \mathrm{Re}\,\rho_{\mathrm{dyn}})\ -\ \tfrac{1}{2}H|\mathrm{Im}\,\rho_{\mathrm{dyn}}|\,\tilde{\epsilon}^{1/2}\hat{\epsilon}^{1/2}\ +\ \mathrm{i}\,H\ \mathrm{Im}\,\rho_{\mathrm{dyn}}\,\big(1-\sqrt{1-\epsilon}\big)\ +\ \mathsf{O}(\hat{\epsilon}^{3/2}\,\tilde{\epsilon}^{1/2})\\ &=\ (M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\ \mathrm{Re}\,\rho_{\mathrm{dyn}})\ -\ \tfrac{1}{2}H|\mathrm{Im}\,\rho_{\mathrm{dyn}}|\,\tilde{\epsilon}^{1/2}\hat{\epsilon}^{1/2}\ +\ \mathrm{i}\tfrac{1}{2}\epsilon H\ \mathrm{Im}\,\rho_{\mathrm{dyn}}\ +\ \mathsf{O}(\hat{\epsilon}^{3/2}\,\tilde{\epsilon}^{1/2})\ +\ \mathsf{O}(\epsilon^2)\,.\end{aligned} \tag{5.28}$$

while

$$\begin{aligned}(\det B)\,\mathsf{s}_{\mathrm{B}}^2\ &=\ \mathrm{tr}\,C\ +\ 4\sqrt{(\mathrm{tr}\,C)^2\ -\ 4(\det A)(\det B)}\\ &=\ (M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\ \mathrm{Re}\,\rho_{\mathrm{dyn}})\ +\ \tfrac{1}{2}H|\mathrm{Im}\,\rho_{\mathrm{dyn}}|\,\tilde{\epsilon}^{1/2}\hat{\epsilon}^{1/2}\ +\ \mathrm{i}\big(1+\sqrt{1-\epsilon}\big)\,H\ \mathrm{Im}\,\rho_{\mathrm{dyn}}\ +\ \mathsf{O}(\hat{\epsilon}^{3/2}\,\tilde{\epsilon}^{1/2})\\ &=\ (M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\ \mathrm{Re}\,\rho_{\mathrm{dyn}})\ +\ \tfrac{1}{2}H|\mathrm{Im}\,\rho_{\mathrm{dyn}}|\,\tilde{\epsilon}^{1/2}\hat{\epsilon}^{1/2}\ +\ \mathrm{i}\big(2-\tfrac{1}{2}\epsilon\big)\,H\ \mathrm{Im}\,\rho_{\mathrm{dyn}}\ +\ \mathsf{O}(\hat{\epsilon}^{3/2}\,\tilde{\epsilon}^{1/2})\ +\ \mathsf{O}(\epsilon^2)\,.\end{aligned} \tag{5.29}$$

From (5.27),

$$\tilde{\epsilon}^{1/2}\hat{\epsilon}^{1/2} < \frac{2}{\sqrt{3}}\tilde{\epsilon}\,.$$

thus $\mathrm{Re}\,\mathsf{s}_{\mathrm{P}}^2 > 0$ if

$$M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\ \mathrm{Re}\,\rho_{\mathrm{dyn}} \ > \ \frac{2}{\sqrt{3}}H|\mathrm{Im}\,\rho_{\mathrm{dyn}}|\tilde{\epsilon}\,.$$

This is true from Assumption (5.11b). This can be seen by rewriting (5.11b) in equivalent forms,

$$\begin{aligned}
& \frac{\det B}{H\,M}\,\rho_a \ > \ \frac{2-\sqrt{3}}{4}\left(\rho_a - 2\alpha\rho_{\mathrm{f}} + \frac{H}{M}\ \mathrm{Re}\,\rho_{\mathrm{dyn}}\right) \\
\Leftrightarrow \quad & \rho_a - 2\alpha\rho_{\mathrm{f}} + \frac{H}{M}\ \mathrm{Re}\,\rho_{\mathrm{dyn}} \ > \ \frac{2}{\sqrt{3}}\left(\rho_a - 2\alpha\rho_{\mathrm{f}} + \frac{H}{M}\ \mathrm{Re}\,\rho_{\mathrm{dyn}} \ - \ 2\frac{\det B}{H\,M}\,\rho_a\right) \\
\Leftrightarrow \quad & M\rho_a \ - 2\,\alpha\,M\rho_{\mathrm{f}} \ + \ H\ \mathrm{Re}\,\rho_{\mathrm{dyn}} \ > \ \frac{2}{\sqrt{3}}\left(M\rho_a \ - 2\,\alpha\,M\rho_{\mathrm{f}} \ + \ H\ \mathrm{Re}\,\rho_{\mathrm{dyn}} \ - \ 2\frac{\det B}{H}\,\rho_a\right) \\
& \qquad = \ \frac{2}{\sqrt{3}}\,\frac{H\,(M\rho_a - 2\alpha M\rho_{\mathrm{f}} + H\ \mathrm{Re}\,\rho_{\mathrm{dyn}}) \ - \ 2(\det B)\,\rho_a}{H^2\,|\mathrm{Im}\,\rho_{\mathrm{dyn}}|}\,H\,|\mathrm{Im}\,\rho_{\mathrm{dyn}}| \\
& \qquad = \ \frac{2}{\sqrt{3}}\,\tilde{\epsilon}\,H\,|\mathrm{Im}\,\rho_{\mathrm{dyn}}|\,.
\end{aligned}$$

□

Remark 6. For $a > 0$, we have

$$\mathrm{Arg}(-1-\mathrm{i}a) = \mathrm{Arg}(1+a) - \pi \quad , \quad \mathrm{Arg}(-1+a) = \mathrm{Arg}(1-a) + \pi\,.$$

Thus

$$\sqrt{-1 \ - \ \mathrm{i}\,\mathfrak{s}\,\hat{\epsilon}} \ = \ (-\mathfrak{s})\,\mathrm{i}\,\sqrt{1 \ + \ \mathrm{i}\,\mathfrak{s}\,\hat{\epsilon}}\,. \tag{5.30}$$

As a result,

$$\sqrt{-1 \ - \ \mathrm{i}a} \ = \ -\mathrm{i}\,\sqrt{1 \ + \ a} \quad , \quad \sqrt{-1 + \mathrm{i}a} \ = \ \mathrm{i}\,\sqrt{1 \ - \ a}\,.$$

5.2 Definition of slowness

We have tested with the materials listed in table 2 for frequencies in the range [1Hz , 1MHz] and viscosity in $[0, 10^{-2}]$ Pa.s, and we have found that the slowness square defined in (4.9) always observes the following equations

$$\mathrm{Re}\,\mathsf{s}_\bullet^2 > 0 \quad , \quad -\mathfrak{s}\ \mathrm{Im}\,\mathsf{s}_\bullet^2 \ > \ 0 \quad , \quad \text{for } \bullet = \mathrm{P},\mathrm{S},\mathrm{B}\,, \tag{5.31}$$

in the presence of viscosity and

$$\mathrm{Re}\,\mathsf{s}_\bullet^2 \ > \ 0 \quad , \quad \mathrm{Im}\,\mathsf{s}_\bullet^2 \ = \ 0 \quad \text{for } \bullet = \mathrm{P},\mathrm{S},\mathrm{B}\,. \tag{5.32}$$

for zero-viscosity.

Definition 1. Under the assumption that the slowness square defined in (4.9) satisfies (5.31), we define the slowness to be

$$\mathsf{s}_\bullet \ := \ -\mathfrak{s}\,\sqrt{\mathsf{s}_\bullet^2}\,. \tag{5.33}$$

Here, the square root uses the principle argument range i.e. the interval $(-\pi, \pi]$.

Under assumption (5.31), the slowness satisfies

$$\mathrm{Im}\,\mathsf{s}_\bullet \ \geq \ 0 \quad , \quad -\mathfrak{s}\ \mathrm{Re}\,\mathsf{s}_\bullet \ \geq \ 0\,. \tag{5.34}$$

Remark 7. Note that this property is guaranteed under the assumption of Prop 4 for zero-viscosity and Prop 5 with viscosity. △

6 Potential method for isotropic poroelastic equations

In this section, we use the form of the poroelastic to find a decomposition of the displacements u and w as functions of scalars unknowns called potentials. Classically, to obtain analytic solutions, fundamental solutions or Green kernel for isotropic elastic or poroelastic equation, one uses the Helmholtz decomposition, the unknowns in this approach are called the Helmholtz potentials. Here we use a slightly different method without imposing the Helmholtz decomposition on the original unknowns. Further discussions on the differences between the two methods are given in Appendix C

6.1 Derivation

Notations We recall the following definitions from (4.10).

$$A(\omega) := \begin{pmatrix} \rho_a & \rho_{\mathrm{f}} \\ \rho_{\mathrm{f}} & \rho_{\mathrm{dyn}}(\omega) \end{pmatrix} \quad , \quad B := \begin{pmatrix} H & \alpha\, M \\ \alpha M & M \end{pmatrix} \quad , \quad B^{\mathrm{cof}} = \begin{pmatrix} M & -\alpha\, M \\ -\alpha M & H \end{pmatrix} \quad , \quad C := B^{\mathrm{cof}} A\,.$$

Recall that $\mathrm{s_P}$ and $\mathrm{s_B}$ defined in (4.9) are the eigenvalues of $B^{-1}A$, *cf.* (4.10), with corresponding eigenvector, *cf.* (4.26)

$$\begin{pmatrix} 1 \\ -\dfrac{H\,\mathrm{s_P^2} - \rho_a}{\mathrm{s_P^2}\,\alpha\, M \;-\; \rho_{\mathrm{f}}} \end{pmatrix} \quad , \quad \begin{pmatrix} 1 \\ -\dfrac{H\,\mathrm{s_B^2} - \rho_a}{\mathrm{s_B^2}\,\alpha\, M \;-\; \rho_{\mathrm{f}}} \end{pmatrix} .$$

Here, the change of basis matrix P is,

$$P(\omega) \;:=\; \begin{pmatrix} 1 & 1 \\ \beta_{\mathrm{P}} & \beta_{\mathrm{B}} \end{pmatrix} \;=\; \begin{pmatrix} 1 & 1 \\ -\frac{H\,\mathrm{s_P^2} - \rho_a}{\mathrm{s_P^2}\,\alpha\,M - \rho_{\mathrm{f}}} & -\frac{H\,\mathrm{s_B^2} - \rho_a}{\mathrm{s_B^2}\,\alpha\,M - \rho_{\mathrm{f}}} \end{pmatrix}, \tag{6.1}$$

with $\beta_\bullet$ defined in (4.13). Using P, we write

$$B^{-1}A(\omega) \;=\; P(\omega) \begin{pmatrix} \mathrm{s_P^2}(\omega) & 0 \\ 0 & \mathrm{s_B^2}(\omega) \end{pmatrix} P^{-1}(\omega)\,. \tag{6.2}$$

We also recall the following identities in 2D for function f and vector, $\mathbf{v}$

$$\nabla\cdot\mathbf{curl} \;=\; 0 \quad , \quad \mathrm{curl}\,\nabla \;=\; 0\,,$$
$$\mathrm{curl}\,\mathbf{curl}\,f \;=\; -\Delta f \quad , \quad \mathbf{curl}\,\mathrm{curl}\,\mathbf{v} \;=\; \nabla\,\nabla\cdot\mathbf{v} \;-\; \Delta\mathbf{v}\,.$$

Proposition 6. *Consider* (u, w) *a pair of solutions to the poroelastic equation* (3.27),

$$-\omega^2\rho_a \mathrm{u} \;-\; \rho_{\mathrm{f}}\omega^2\mathrm{w} - H\nabla\nabla\cdot\mathrm{u} + \mu_{\mathrm{fr}}\,\mathbf{curl}\,\mathrm{curl}\,\mathrm{u} - \alpha M\nabla\nabla\cdot\mathrm{w} = \mathbf{f}\,, \tag{6.3a}$$

$$-\omega^2\rho_{\mathrm{f}}\mathrm{u} \;-\; \omega^2\,\rho_{dyn}(\omega)\,\mathrm{w} - \mathfrak{m}\,\nabla\nabla\cdot\mathrm{w} \;-\; M\alpha\nabla\nabla\cdot\mathrm{u} = \tilde{\mathbf{f}}\,. \tag{6.3b}$$

Then they have to be of the form,

$$\begin{aligned} \omega^2\mathrm{u} \;&=\; -\mathrm{s_P^{-2}}\,\nabla\chi_{\mathrm{P}} \;-\; \mathrm{s_B^{-2}}\,\nabla\chi_{\mathrm{B}} \;+\; \mathrm{s_S^{-2}}\,\mathbf{curl}\,\chi_{\mathrm{S}} \;-\; \pi_1\left(A^{-1}\begin{pmatrix}\mathbf{f}\\ \tilde{\mathbf{f}}\end{pmatrix}\right), \\ \omega^2\mathrm{w} \;&=\; -\frac{\beta_{\mathrm{P}}}{\mathrm{s_P^2}}\nabla\chi_{\mathrm{P}} \;-\; \frac{\beta_{\mathrm{B}}}{\mathrm{s_B^2}}\nabla\chi_{\mathrm{B}} \;-\; \frac{\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}}}{\det A}\mathbf{curl}\,\chi_{\mathrm{S}} \;-\; \pi_2\left(A^{-1}\begin{pmatrix}\mathbf{f}\\ \tilde{\mathbf{f}}\end{pmatrix}\right). \end{aligned} \tag{6.4}$$

Here for $i \;=\; 1,2$, π_i *is the projection onto the* i*-th component of a vector, and the potential* $\chi_\bullet$ *with*

- $\bullet = \mathrm{P},\mathrm{S},\mathrm{B}$ *satisfy the Helmholtz equation*

$$\big(-\Delta \;-\; \omega^2\, \mathrm{s}_{\mathrm{P}}^2\big)\,\chi_{\mathrm{P}} \;=\; \pi_1 \left(P^{-1}\,B^{-1}\begin{pmatrix}\nabla\cdot\mathbf{f}\\ \nabla\cdot\tilde{\mathbf{f}}\end{pmatrix}\right), \tag{6.5a}$$

$$\big(-\Delta \;-\; \omega^2\, \mathrm{s}_{\mathrm{B}}^2\big)\,\chi_{\mathrm{B}} \;=\; \pi_2 \left(P^{-1}\,B^{-1}\begin{pmatrix}\nabla\cdot\mathbf{f}\\ \nabla\cdot\tilde{\mathbf{f}}\end{pmatrix}\right). \tag{6.5b}$$

and

$$\big(-\Delta \;-\; \omega^2\, \mathrm{s}_{\mathrm{S}}^2\big)\,\chi_{\mathrm{S}} \;=\; \mathrm{s}_{\mathrm{S}}^2\;\pi_1 \left(A^{-1}\begin{pmatrix}\mathrm{curl}\mathbf{f}\\ \mathrm{curl}\tilde{\mathbf{f}}\end{pmatrix}\right). \tag{6.6}$$

Proof. As unknowns, we will work with,

$$\begin{aligned}\varphi &:= \nabla\cdot\mathrm{u} \quad , \quad \tilde{\varphi} := \nabla\cdot\mathrm{w}\,,\\ \psi &:= \mathrm{curl}\,\mathrm{u} \quad , \quad \tilde{\psi} := \mathrm{curl}\,\mathrm{w}\,.\end{aligned} \tag{6.7}$$

Step 1 We first obtain a system of equations in terms of φ, $\tilde{\varphi}$, ψ and $\tilde{\psi}$. The first two equations are obtained by taking $\nabla\cdot$ of the equations (6.3). Using $\nabla\cdot\mathbf{curl} = 0$, and $\nabla\cdot\nabla = \Delta$, $\nabla\cdot$ of equation (6.3a) gives

$$\nabla\cdot\big(-\omega^2\rho_a\mathrm{u} \;-\; \rho_{\mathrm{f}}\omega^2\mathrm{w} - H\nabla\nabla\cdot\mathrm{u} + \mu_{\mathrm{fr}}\,\mathbf{curl}\,\mathrm{curl}\,\mathrm{u} - \alpha M\nabla\nabla\cdot\mathrm{w}\big) \;=\; \nabla\cdot\mathbf{f}\,,$$

$$\Rightarrow\quad -\omega^2\rho_a\varphi \;-\; \rho_{\mathrm{f}}\omega^2\tilde{\varphi} - H\Delta\varphi - \alpha M\Delta\tilde{\varphi} \;=\; \nabla\cdot\mathbf{f}\,,$$

and $\nabla\cdot$ of equation (6.3b) leads to

$$\nabla\cdot\big(-\omega^2\rho_{\mathrm{f}}\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{dyn}}(\omega)\,\mathrm{w} - M\,\nabla\nabla\cdot\mathrm{w} \;-\; M\alpha\nabla\nabla\cdot\mathrm{u}\big) \;=\; \nabla\cdot\tilde{\mathbf{f}}\,,$$

$$\Rightarrow\quad -\omega^2\rho_{\mathrm{f}}\varphi \;-\; \omega^2\,\rho_{\mathrm{dyn}}(\omega)\,\tilde{\varphi} - M\,\Delta\tilde{\varphi} \;-\; M\alpha\Delta\varphi \;=\; \nabla\cdot\tilde{\mathbf{f}}\,.$$

The third and fourth equations are obtained by taking curl of equations (6.3). Using $\mathrm{curl}\,\mathbf{curl} = -\Delta$ and $\mathrm{curl}\nabla = 0$, (6.3a) gives

$$\mathrm{curl}\big(-\omega^2\rho_a\mathrm{u} \;-\; \rho_{\mathrm{f}}\omega^2\mathrm{w} \;-\; H\nabla\nabla\cdot\mathrm{u} \;+\; \mu_{\mathrm{fr}}\,\mathbf{curl}\,\mathrm{curl}\,\mathrm{u} \;-\; \alpha\,M\,\nabla\,\nabla\cdot\mathrm{w}\big) \;=\; \mathrm{curl}\,\mathbf{f}\,,$$

$$\Rightarrow\quad -\omega^2\,\rho_a\,\psi \;-\; \rho_{\mathrm{f}}\,\omega^2\,\tilde{\psi} \;-\; \mu_{\mathrm{fr}}\,\Delta\psi \;=\; \mathrm{curl}\,\mathbf{f}\,,$$

while the second equation (6.3b) gives

$$\mathrm{curl}\big(-\omega^2\rho_{\mathrm{f}}\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{dyn}}(\omega)\,\mathrm{w} - M\,\nabla\nabla\cdot\mathrm{w} \;-\; M\alpha\nabla\nabla\cdot\mathrm{u}\big) \;=\; \mathrm{curl}\,\tilde{\mathbf{f}}\,,$$

$$\Rightarrow\quad -\omega^2\,\rho_{\mathrm{f}}\,\psi \;-\; \omega^2\,\rho_{\mathrm{dyn}}(\omega)\,\tilde{\psi} \;=\; \mathrm{curl}\,\tilde{\mathbf{f}}\,.$$

We rewrite these four equations in matrix form to obtain,

$$-\omega^2\,A(\omega)\begin{pmatrix}\varphi\\ \tilde{\varphi}\end{pmatrix} \;-\; B\Delta\begin{pmatrix}\varphi\\ \tilde{\varphi}\end{pmatrix} \;=\; \begin{pmatrix}\nabla\cdot\mathbf{f}\\ \nabla\cdot\tilde{\mathbf{f}}\end{pmatrix}, \tag{6.8a}$$

$$\text{and}\quad -\,\omega^2\,A(\omega)\begin{pmatrix}\psi\\ \tilde{\psi}\end{pmatrix} \;-\; \begin{pmatrix}\mu_{\mathrm{fr}} & 0\\ 0 & 0\end{pmatrix}\Delta\begin{pmatrix}\psi\\ \tilde{\psi}\end{pmatrix} \;=\; \begin{pmatrix}\mathrm{curl}\,\mathbf{f}\\ \mathrm{curl}\,\tilde{\mathbf{f}}\end{pmatrix}. \tag{6.8b}$$

Step 2a Multiply by A^{-1} on both sides, we first rewrite (6.8b) as

$$-\omega^2\begin{pmatrix}\psi\\ \tilde{\psi}\end{pmatrix} \;-\; A^{-1}\begin{pmatrix}\mu_{\mathrm{fr}} & 0\\ 0 & 0\end{pmatrix}\Delta\begin{pmatrix}\psi\\ \tilde{\psi}\end{pmatrix} \;=\; A^{-1}\begin{pmatrix}\mathrm{curl}\,\mathbf{f}\\ \mathrm{curl}\,\tilde{\mathbf{f}}\end{pmatrix}. \tag{6.9}$$

Using the identity

$$A^{-1}\begin{pmatrix}\mu_{\mathrm{fr}} & 0\\ 0 & 0\end{pmatrix} = \frac{1}{\det A}\begin{pmatrix}\rho_{\mathrm{dyn}} & -\rho_{\mathrm{f}}\\ -\rho_{\mathrm{f}} & \rho_a\end{pmatrix}\begin{pmatrix}\mu_{\mathrm{fr}} & 0\\ 0 & 0\end{pmatrix} = \frac{1}{\det A}\begin{pmatrix}\rho_{\mathrm{dyn}}\,\mu_{\mathrm{fr}} & 0\\ -\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}} & 0\end{pmatrix}, \tag{6.10}$$

the first component of (6.9) gives,

$$-\omega^2 \frac{\det A}{\rho_{\text{dyn}}\,\mu_{\text{fr}}}\psi - \Delta\psi = \frac{\det A}{\rho_{\text{dyn}}\,\mu_{\text{fr}}}\,\pi_1\left(A^{-1}\begin{pmatrix}\text{curl}\mathbf{f}\\ \text{curl}\tilde{\mathbf{f}}\end{pmatrix}\right).$$

Here π_i for $i = 1, 2$ is the projection onto the i-th component of a vector. Rewriting this in terms of the shear slowness $\mathrm{s_S}$ (4.9a), we obtain that the potential $\psi = \chi_{\mathrm{S}}$ solves the Helmholtz equation (6.6).

Step 2b Apply to both sides of (6.8a) B^{-1}, we obtain

$$-\omega^2\, B^{-1}A(\omega)\begin{pmatrix}\varphi\\ \tilde{\varphi}\end{pmatrix} \;-\; \Delta\begin{pmatrix}\varphi\\ \tilde{\varphi}\end{pmatrix} \;=\; B^{-1}\begin{pmatrix}\nabla\cdot\mathbf{f}\\ \nabla\cdot\tilde{\mathbf{f}}\end{pmatrix}.$$

Next, using the diagonalizing form (6.2) of $B^{-1}A$, the above equation is rewritten as,

$$-\omega^2\, P\begin{pmatrix}\mathrm{s_P^2} & 0\\ 0 & \mathrm{s_B^2}\end{pmatrix} P^{-1}\begin{pmatrix}\varphi\\ \tilde{\varphi}\end{pmatrix} \;-\; \Delta\begin{pmatrix}\varphi\\ \tilde{\varphi}\end{pmatrix} \;=\; B^{-1}\begin{pmatrix}\nabla\cdot\mathbf{f}\\ \nabla\cdot\tilde{\mathbf{f}}\end{pmatrix},$$

$$\Rightarrow -\omega^2 \begin{pmatrix}\mathrm{s_P^2} & 0\\ 0 & \mathrm{s_B^2}\end{pmatrix} P^{-1}\begin{pmatrix}\varphi\\ \tilde{\varphi}\end{pmatrix} \;-\; \Delta P^{-1}\begin{pmatrix}\varphi\\ \tilde{\varphi}\end{pmatrix} \;=\; P^{-1}\,B^{-1}\begin{pmatrix}\nabla\cdot\mathbf{f}\\ \nabla\cdot\tilde{\mathbf{f}}\end{pmatrix}.$$

Define

$$\begin{pmatrix}\chi_{\mathrm{P}}\\ \chi_{\mathrm{B}}\end{pmatrix} \;:=\; P^{-1}\begin{pmatrix}\varphi\\ \tilde{\varphi}\end{pmatrix}. \tag{6.11}$$

Then χ_{P} and χ_{B} satisfy the Helmholtz equations (6.5).

Step 3 We now rewrite u and w in terms of the potential $\chi_\bullet$ which are solutions of the Helmholtz equation (6.5) and (6.6). In particular, from (6.3), we obtain

$$-\omega^2\begin{pmatrix}\mathrm{u}\\ \mathrm{w}\end{pmatrix} \;-\; A^{-1}B\begin{pmatrix}\nabla\varphi\\ \nabla\tilde{\varphi}\end{pmatrix} \;+\; A^{-1}\begin{pmatrix}\mu_{\text{fr}} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{curl}\,\psi\\ 0\end{pmatrix} = A^{-1}\begin{pmatrix}\mathbf{f}\\ \tilde{\mathbf{f}}\end{pmatrix}.$$

Using the diagonalizing form (6.2) and (6.10), this is further written as,

$$-\omega^2\begin{pmatrix}\mathrm{u}\\ \mathrm{w}\end{pmatrix} \;-\; P\begin{pmatrix}\mathrm{s_P^{-2}} & 0\\ 0 & \mathrm{s_B^{-2}}\end{pmatrix}P^{-1}\begin{pmatrix}\nabla\varphi\\ \nabla\tilde{\varphi}\end{pmatrix} \;+\; \frac{1}{\det A}\begin{pmatrix}\rho_{\text{dyn}}\,\mu_{\text{fr}} & 0\\ -\rho_{\text{f}}\,\mu_{\text{fr}} & 0\end{pmatrix}\begin{pmatrix}\mathbf{curl}\,\psi\\ 0\end{pmatrix} = A^{-1}\begin{pmatrix}\mathbf{f}\\ \tilde{\mathbf{f}}\end{pmatrix}.$$

Component wise, the above system gives

$$\omega^2\mathrm{u} \;=\; -\frac{P_{11}}{\mathrm{s_P^2}}\nabla\chi_{\mathrm{P}} \;-\; \frac{P_{12}}{\mathrm{s_B^2}}\nabla\chi_{\mathrm{B}} \;+\; \frac{1}{\mathrm{s_S^2}}\,\mathbf{curl}\,\chi_{\mathrm{S}} \;-\; \pi_1\left(A^{-1}\begin{pmatrix}\mathbf{f}\\ \tilde{\mathbf{f}}\end{pmatrix}\right),$$

$$\omega^2\mathrm{w} \;=\; -\frac{P_{21}}{\mathrm{s_P^2}}\nabla\chi_{\mathrm{P}} \;-\; \frac{P_{22}}{\mathrm{s_B^2}}\nabla\chi_{\mathrm{B}} \;-\; \frac{\rho_{\text{f}}\,\mu_{\text{fr}}}{\det A}\mathbf{curl}\,\chi_{\mathrm{S}} \;-\; \pi_2\left(A^{-1}\begin{pmatrix}\mathbf{f}\\ \tilde{\mathbf{f}}\end{pmatrix}\right).$$

This simplifies to give the final form (6.4) of the displacements.

□

Potential form for unknowns (3.28) **in first order formulation** (3.30)

$$\boxed{\begin{aligned} \mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u} \;&=\; \mathrm{s_P^{-2}}\,\nabla\chi_{\mathrm{P}} \;+\; \mathrm{s_B^{-2}}\,\nabla\chi_{\mathrm{B}} \;-\; \mathrm{s_S^{-2}}\,\mathbf{curl}\,\chi_{\mathrm{S}} \;+\; F\,,\\ \mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{w} \;&=\; \frac{\upbeta_{\mathrm{P}}}{\mathrm{s_P^2}}\nabla\chi_{\mathrm{P}} \;+\; \frac{\upbeta_{\mathrm{B}}}{\mathrm{s_B^2}}\nabla\chi_{\mathrm{B}} \;+\; \frac{\rho_{\text{f}}\,\mu_{\text{fr}}}{\det A}\mathbf{curl}\,\chi_{\mathrm{S}} \;+\; \tilde{F}\,. \end{aligned}} \tag{6.12}$$

To obtain the fluid pressure, we use (note that $\nabla\cdot u = \varphi$ and $\nabla\cdot w = \tilde{\varphi}$),

$$\begin{aligned} \mathrm{p} \;&=\; -M\,\tilde{\varphi} \;-\; M\,\alpha\,\varphi \;-\; M\mathbf{f}_{\mathrm{p}} \\ &=\; -M\left(\beta_{\mathrm{P}}\,\chi_{\mathrm{P}} \;+\; \beta_{\mathrm{B}}\,\chi_{\mathrm{B}}\right) \;-\; M\,\alpha\left(\chi_{\mathrm{P}}+\chi_{\mathrm{B}}\right) \;-\; M\,\mathbf{f}_{\mathrm{p}} \\ &=\; -M\left(\beta_{\mathrm{P}} \;+\; \alpha\right)\chi_{\mathrm{P}} \;-\; M\left(\beta_{\mathrm{B}} \;+\; \alpha\right)\chi_{\mathrm{B}} \;-\; M\,\mathbf{f}_{\mathrm{p}}\,. \end{aligned}$$

Thus

$$\boxed{\mathrm{p} \;=\; -M\left(\beta_{\mathrm{P}} \;+\; \alpha\right)\chi_{\mathrm{P}} \;-\; M\left(\beta_{\mathrm{B}} \;+\; \alpha\right)\chi_{\mathrm{B}} \;-\; M\,\mathbf{f}_{\mathrm{p}}\,.} \tag{6.13}$$

To find $\boldsymbol{\tau}$ we use

$$\begin{aligned} \boldsymbol{\tau} \;&=\; \mu_{\mathrm{fr}}\left(\nabla\mathrm{u}+(\nabla\mathrm{u})^{t}\right) \;+\; \left(-\tfrac{2}{3}\mu_{\mathrm{fr}} \;+\; \mathrm{k}_{\mathrm{fr}} \;+\; M\alpha^{2}\right)\,\nabla\cdot\mathrm{u}\;\mathbf{Id} \;+\; \alpha\,M\,\nabla\cdot\mathrm{w}\;\mathbf{Id} \\ \Rightarrow\quad \omega^{2}\,\boldsymbol{\tau} \;&=\; \mu_{\mathrm{fr}}\left(\nabla\omega^{2}\,\mathrm{u}+(\nabla\omega^{2}\,\mathrm{u})^{t}\right) \;+\; \left(-\tfrac{2}{3}\mu_{\mathrm{fr}} \;+\; \mathrm{k}_{\mathrm{fr}} \;+\; M\alpha^{2}\right)\,\omega^{2}\nabla\cdot\mathrm{u}\;\mathbf{Id} \;+\; \alpha\,M\,\omega^{2}\,\nabla\cdot\mathrm{w}\;\mathbf{Id}. \end{aligned}$$

After simplification, we obtain

$$\boxed{\begin{aligned} \omega^{2}\,\boldsymbol{\tau} \;=\; \mu_{\mathrm{fr}}&\left(-\frac{2}{\mathrm{s}_{\mathrm{P}}^{2}}\,\nabla^{2}\chi_{\mathrm{P}} \;-\; \frac{2}{\mathrm{s}_{\mathrm{B}}^{2}}\,\nabla^{2}\chi_{\mathrm{B}} \;+\; \frac{\nabla\mathbf{curl}\,\chi_{\mathrm{S}} \;+\; (\nabla\mathbf{curl}\,\chi_{\mathrm{S}})^{t}}{\mathrm{s}_{\mathrm{S}}^{2}} \;-\; \left(\nabla F+(\nabla F)^{t}\right)\right) \\ &+\; \omega^{2}\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} \;+\; \mathrm{k}_{\mathrm{fr}} \;+\; M\alpha^{2}\right)\,\left(\chi_{\mathrm{P}} \;+\; \chi_{\mathrm{B}}\right)\;\mathbf{Id} \\ &+\; \omega^{2}\,\alpha\,M\,\left(\beta_{\mathrm{P}}\,\chi_{\mathrm{P}} \;+\; \beta_{\mathrm{B}}\,\chi_{\mathrm{B}}\right)\;\mathbf{Id}\,. \end{aligned}} \tag{6.14}$$

6.2 Expansion of generic solutions to homogeneous equations in terms of Bessel functions

Here, we obtain the form for a general solution in terms of Bessel functions to the homogeneous poroelastic equation in three types of domains: on a disc, in an annulus and outside of a disc. When there are no source, i.e. all sources are zero in (6.6) and (6.5), then the $\chi_{\bullet}$ satisfy the homogeneous Helmholtz equation:

$$\begin{aligned} \left(-\Delta \;-\; \omega^{2}\,\mathsf{s}_{\mathrm{S}}^{2}\right)\chi_{\mathrm{S}} \;&=\; 0\,, \\ \left(-\Delta \;-\; \omega^{2}\,\mathsf{s}_{\mathrm{P}}^{2}\right)\chi_{\mathrm{P}} \;&=\; 0\,, \\ \left(-\Delta \;-\; \omega^{2}\,\mathsf{s}_{\mathrm{B}}^{2}\right)\chi_{\mathrm{B}} \;&=\; 0\,. \end{aligned} \tag{6.15}$$

On each considered domain, χ can be given as an expansion in terms of Bessel functions.

a) On a disc $\mathbb{B}_{\mathfrak{a}}$ centered at the origin and of radius $\mathfrak{a}$, a generic solution is given by:

$$\chi_{\bullet}(\boldsymbol{x}) \;=\; \sum_{k\in\mathbb{Z}} a_{\bullet,k}\,\mathrm{J}_{k}(\,\omega\,\mathsf{s}_{\bullet}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \quad,\quad \bullet\in\{\mathrm{S},\mathrm{P},\mathrm{B}\,.\} \tag{6.16}$$

b) An outgoing solution on $\mathbb{R}^{2}\setminus\mathbb{B}_{\mathfrak{a}}$ is given by

$$\chi_{\bullet}(\boldsymbol{x}) \;=\; \sum_{k\in\mathbb{Z}} a_{\bullet,k}\,\mathrm{H}_{k}^{(1)}(\,\omega\,\mathsf{s}_{\bullet}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \quad,\quad \bullet\in\{\mathrm{S},\mathrm{P},\mathrm{B}\,.\} \tag{6.17}$$

See also Remark 8 regarding the 'outgoing'-ness of this solution.

c) On an annulus between inner radius $\mathfrak{a}$ and outer radius $\mathfrak{b}$, a generic solution is given by:

$$\chi_{\bullet}(\boldsymbol{x}) \;=\; \sum_{k\in\mathbb{Z}} a_{\bullet,k}\,\mathrm{H}_{k}^{(1)}(\omega\,\mathsf{s}_{\bullet}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}} \tilde{a}_{\bullet,k}\,\mathrm{H}_{k}^{(2)}(\omega\,\mathsf{s}_{\bullet}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \quad,\quad \bullet\in\{\mathrm{S},\mathrm{P},\mathrm{B}\,.\} \tag{6.18}$$

To obtain the expansion of $\mathbf{u}$, $\mathbf{w}$, $\boldsymbol{\tau}$ and p, depending on the domain, it remains to substitute the expression for $\chi_{\bullet}$ (6.16), (6.17) or (6.18) into (6.12) - (6.14). We write the expansion for the case a (on a disc) and case b (outgoing). Denote by Z_{k} a Bessel function[2].

[2] We recall the definitions of Bessel and Hankel functions.

For case a, $\mathrm{Z}_k = \mathrm{J}_k$, and for case b, $\mathrm{Z}_k = \mathrm{H}_k^{(1)}$. Calculations details are given in appendix A.

$$
\begin{aligned}
\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u} \;=\; & \sum_{k\in\mathbb{Z}} a_k\;\mathsf{s}_{\mathrm{P}}^{-1}\,\omega\,\mathrm{Z}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\mathbf{e}_r \;+\; \sum_{k\in\mathbb{Z}} a_k\;\mathsf{s}_{\mathrm{P}}^{-2}\,\frac{\mathrm{i}k}{|\boldsymbol{x}|}\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\mathbf{e}_\theta \\
& + \sum_{k\in\mathbb{Z}} b_k\;\mathsf{s}_{\mathrm{B}}^{-1}\,\omega\,\mathrm{Z}_k'(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\mathbf{e}_r \;+\; \sum_{k\in\mathbb{Z}} b_k\;\mathsf{s}_{\mathrm{B}}^{-2}\,\frac{\mathrm{i}k}{|\boldsymbol{x}|}\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\mathbf{e}_\theta \\
& - \sum_{k\in\mathbb{Z}} c_k\;\mathsf{s}_{\mathrm{S}}^{-2}\,\frac{\mathrm{i}k}{|\boldsymbol{x}|}\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\mathbf{e}_r \;+\; \sum_{k\in\mathbb{Z}} c_k\;\mathsf{s}_{\mathrm{S}}^{-1}\,\omega\,\mathrm{Z}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\mathbf{e}_\theta\,,
\end{aligned}
\tag{6.19}
$$

$$
\begin{aligned}
\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{w} \;=\; & \sum_{k\in\mathbb{Z}} a_k\;\frac{\beta_{\mathrm{P}}}{\mathsf{s}_{\mathrm{P}}}\,\omega\,\mathrm{Z}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\mathbf{e}_r \;+\; \sum_{k\in\mathbb{Z}} a_k\;\frac{\beta_{\mathrm{P}}}{\mathsf{s}_{\mathrm{P}}^2}\,\frac{\mathrm{i}k}{|\boldsymbol{x}|}\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\mathbf{e}_\theta \\
& + \sum_{k\in\mathbb{Z}} b_k\;\frac{\beta_{\mathrm{B}}}{\mathsf{s}_{\mathrm{B}}}\,,\omega\,\mathrm{Z}_k'(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\mathbf{e}_r \;+\; \sum_{k\in\mathbb{Z}} b_k\;\frac{\beta_{\mathrm{B}}}{\mathsf{s}_{\mathrm{B}}^2}\,\frac{\mathrm{i}k}{|\boldsymbol{x}|}\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\mathbf{e}_\theta \\
& + \sum_{k\in\mathbb{Z}} c_k\;\frac{\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}}}{\det A}\,\frac{\mathrm{i}k}{|\boldsymbol{x}|}\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\mathbf{e}_r \;-\; \sum_{k\in\mathbb{Z}} c_k\;\frac{\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}}}{\det A}\,\omega\,\mathrm{Z}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\mathbf{e}_\theta\,,
\end{aligned}
\tag{6.20}
$$

In polar basis,

$$
\boldsymbol{\tau} = \boldsymbol{\tau}_{rr}\,\mathbf{e}_r\otimes\mathbf{e}_r + \boldsymbol{\tau}_{r\theta}\,\mathbf{e}_r\otimes\mathbf{e}_\theta + \boldsymbol{\tau}_{\theta r}\,\mathbf{e}_\theta\otimes\mathbf{e}_r + \boldsymbol{\tau}_{\theta\theta}\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta. \tag{6.21}
$$

Note that $\boldsymbol{\tau}$ is symmetric, $\boldsymbol{\tau}_{r\theta} = \boldsymbol{\tau}_{\theta r}$. Here we list only the two most important components,

$$
\begin{aligned}
\omega^2\boldsymbol{\tau}_{rr} = & -\sum_{k\in\mathbb{Z}} \frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|}\,a_k\,\mathrm{Z}_{k+1}(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}} \frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{P}}^2\,|\boldsymbol{x}|^2}\,a_k\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \\
& + \sum_{k\in\mathbb{Z}} 2\,\mu_{\mathrm{fr}}\,a_k\,\omega^2\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} - \sum_{k\in\mathbb{Z}} \frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{P}}^2\,|\boldsymbol{x}|^2}\,a_k\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \\
& - \sum_{k\in\mathbb{Z}} \frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|}\,b_k\,\mathrm{Z}_{k+1}(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}} \frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{B}}^2\,|\boldsymbol{x}|^2}\,b_k\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \\
& + \sum_{k\in\mathbb{Z}} 2\,\mu_{\mathrm{fr}}\,b_k\,\omega^2\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} - \sum_{k\in\mathbb{Z}} \frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{B}}^2\,|\boldsymbol{x}|^2}\,b_k\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \\
& + \sum_{k\in\mathbb{Z}} \frac{2\,\mu_{\mathrm{fr}}}{\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|}\;c_k\,\omega\,\mathsf{s}_{\mathrm{S}}\,\mathrm{i}k\,\mathrm{Z}_k'(\omega\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \\
& + \sum_{k\in\mathbb{Z}} \omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\,M\beta_{\mathrm{P}}\right)\;a_k\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \\
& + \sum_{k\in\mathbb{Z}} \omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\,M\beta_{\mathrm{B}}\right)\,b_k\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\,,
\end{aligned}
\tag{6.22}
$$

Bessel functions are solutions of: $\quad z^2\dfrac{d^2w}{dz^2} + z\dfrac{dw}{dz} + (z^2-\nu^2)w = 0.$

First-order Bessel function: $\quad J_\nu = (\frac{1}{2}z)^\nu \sum_{k=0}^{\infty}(-1)^k\dfrac{(\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)}\quad$ with ν the mode and $\quad \Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt\,.$

Second-order Bessel function:

$$
Y_\nu = \frac{J_\nu(z)cos(\nu\pi) - J_{-\nu}(z)}{sin(\nu\pi)}
$$

Two kind of Hankel functions are expressed: $H_\nu^{(1)} = J_\nu(\boldsymbol{x}) + iY_\nu(\boldsymbol{x})$, and $H_\nu^{(2)} = J_\nu(\boldsymbol{x}) - iY_\nu(\boldsymbol{x})$.

and

$$\begin{aligned}
\omega^2\,\boldsymbol{\tau}_{r\theta} \;=\; & -\sum_{k\in\mathbb{Z}} \frac{2\,\mu_{\rm fr}\omega\,\mathrm{i}\,k}{|\boldsymbol{x}|\,\mathsf{s}_{\rm P}}\, a_k\,\mathrm{Z}_k'(\omega\,\mathsf{s}_{\rm P}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}} \frac{2\,\mathrm{i}\,\mu_{\rm fr}\,k}{|\boldsymbol{x}|^2\,\mathsf{s}_{\rm P}^2}\, a_k\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\rm P}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\\
& -\sum_{k\in\mathbb{Z}} \frac{2\,\mu_{\rm fr}\omega\,\mathrm{i}\,k}{|\boldsymbol{x}|\,\mathsf{s}_{\rm B}}\, b_k\,\mathrm{Z}_k'(\omega\,\mathsf{s}_{\rm B}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}} \frac{2\,\mathrm{i}\,\mu_{\rm fr}\,k}{|\boldsymbol{x}|^2\,\mathsf{s}_{\rm B}^2}\, b_k\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\rm B}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\\
& -\sum_{k\in\mathbb{Z}} \frac{\mu_{\rm fr}\,k^2}{|\boldsymbol{x}|^2\,\mathsf{s}_{\rm S}^2}\, c_k\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\rm S}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}} \frac{\mu_{\rm fr}\,\omega}{|\boldsymbol{x}|\,\mathsf{s}_{\rm S}}\, c_k\,\mathrm{Z}_k'(\omega\,\mathsf{s}_{\rm S}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\\
& -\sum_{k\in\mathbb{Z}} \mu_{\rm fr}\,\frac{\omega}{\mathsf{s}_{\rm S}\,|\boldsymbol{x}|}\, c_k\,\mathrm{Z}_{k+1}(\omega\,\mathsf{s}_{\rm S}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}} \mu_{\rm fr}\,\frac{k}{\mathsf{s}_{\rm S}^2\,|\boldsymbol{x}|^2} c_k\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\rm S}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\\
& +\sum_{k\in\mathbb{Z}} \mu_{\rm fr}\,\omega^2\,c_k\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\rm S}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} - \sum_{k\in\mathbb{Z}} \mu_{\rm fr}\,\frac{k^2}{\mathsf{s}_{\rm S}^2\,|\boldsymbol{x}|^2}\, c_k\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\rm S}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}.
\end{aligned} \tag{6.23}$$

Finally,

$$\mathrm{p} \;=\; -\sum_{k\in\mathbb{Z}} a_k\,M\,\big(\upbeta_{\rm P} \;+\; \alpha\big)\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\rm P}\,|\boldsymbol{x}|) \;-\; \sum_{k\in\mathbb{Z}} b_k\,M\,\big(\upbeta_{\rm B} \;+\; \alpha\big)\,\mathrm{Z}_k(\omega\,\mathsf{s}_{\rm P}\,|\boldsymbol{x}|)\,. \tag{6.24}$$

6.3 Notion of outgoing solution

As a corollary of the form of solution given in (6.12) in terms of the potentials which are solutions of the Helmholtz equation, we can formulate a definition of outgoing solution for poroelasticity. This generalizes the Kupradze radiation condition for isotropic elasticity *cf.* [24], see also *e.g.* [, 2.1e] in 3D or *e.g.* [28, Eqn 4] in 2D. We recall the form of the solutions given in (6.12), we can write

$$\begin{pmatrix}\mathbf{u}\\ \mathbf{w}\end{pmatrix} = P\,\frac{1}{\mathfrak{s}\mathrm{i}\omega}\begin{pmatrix}\frac{\nabla\chi_P}{\mathsf{s}_{\rm P}^2}\\[4pt] \frac{\nabla\chi_B}{\mathsf{s}_{\rm B}^2}\end{pmatrix} + \frac{1}{\mathfrak{s}\mathrm{i}\omega}\,\mathrm{curl}\,\chi_S\begin{pmatrix}-\frac{1}{\mathsf{s}_{\rm S}^2}\\[4pt] \frac{\rho_{\rm f}\mu_{\rm fr}}{\det A}\end{pmatrix}, \tag{6.25}$$

where P is the matrix defined in (6.1) and the potentials $\chi_\bullet$, $\bullet = \mathrm{P},\mathrm{B},\mathrm{S}$ satisfy the Helmholtz equations

$$\begin{aligned}
\big(-\Delta \;-\; \omega^2\,\mathsf{s}_{\rm S}^2\big)\,\chi_{\rm S} \;&=\; 0\,,\\
\big(-\Delta \;-\; \omega^2\,\mathsf{s}_{\rm P}^2\big)\,\chi_{\rm P} \;&=\; 0\,,\\
\big(-\Delta \;-\; \omega^2\,\mathsf{s}_{\rm B}^2\big)\,\chi_{\rm B} \;&=\; 0\,.
\end{aligned} \tag{6.26}$$

The notions of outgoing solution $\mathbf{u}$ and $\mathbf{w}$ are based on that imposed on $\chi_\bullet$, i.e. the Sommerfeld radiation condition for Helmholtz equation. Using the slowness defined in Definition 1, we define the wavenumber

$$\mathsf{k}_\bullet = \omega\,\mathsf{s}_\bullet\,. \tag{6.27}$$

Under the assumption (5.31), from the property of slowness in (5.34), the wavenumber thus has the property

$$\mathrm{Im}\,\mathsf{k}_\bullet \;\geq\; 0 \quad , \quad -\mathfrak{s}\;\mathrm{Re}\,\mathsf{k}_\bullet \;\geq\; 0\,. \tag{6.28}$$

Remark 8. These properties guarantee that when we use $\mathrm{H}_k^{(1)}$ to describe the potentials in (6.17), the resulting solution given by $\mathrm{H}_0^{(1)}$, is outgoing in the case without viscosity, and decreases and L^2 in the presence of viscosity (in both convention). Here we follow the outgoing convention discussed in Appendix G in [3]. In particular,

$$\mathrm{H}_k^{(1)}(\,\mathsf{k}_\bullet\,|\boldsymbol{x}|) \;\sim\; e^{\mathrm{i}\mathsf{k}_\bullet\,|\boldsymbol{x}|} \;=\; e^{\mathrm{i}\,(\mathrm{Re}\,\mathsf{k}_\bullet)\,|\boldsymbol{x}|}\;e^{-(\mathrm{Im}\,\mathsf{k}_\bullet)\,|\boldsymbol{x}|}\,.$$

△

For $\bullet = \mathrm{P},\mathrm{B},\mathrm{S}$, $\chi_\bullet$ is called $\mathsf{k}_\bullet$-outgoing if it satisfies the Sommerfeld radiation condition at wave number $\mathsf{k}_\bullet$ uniformly,

$$\lim_{r\to\infty}\sqrt{r}\,\left(\frac{\partial\varphi_\bullet}{\partial r} \;-\; \mathrm{i}\,\mathsf{k}_\bullet\,\varphi_\bullet\right) \;=\; 0\,. \tag{6.29}$$

Using the identity, $\nabla\cdot\mathbf{curl} = \;0$ and $\mathrm{curl}\,\nabla \;=\; 0$ in equation (6.29), we propose the following definition for the outgoing solution for isotropic poroelastic equation. Note that the strain $\boldsymbol{\tau}$ and the pressure p are uniquely determined by the displacements / velocity. It suffices to impose outgoing criteria for $\mathbf{u}$ and $\mathbf{w}$ (if using the first order formulation) or u and w (if using the original equation).

Definition 2 (Outgoing solutions). The fields $\mathbf{u}$ and $\mathbf{w}$ are called outgoing solutions of the poroelastic equations (3.30) if they satisfy the following radiations conditions.

1. Their rotational $\operatorname{curl}\mathbf{u}$ and $\operatorname{curl}\mathbf{w}$ satisfy the outgoing Sommerfeld radiation condition with wavenumber k_S, *i.e.* for $\varphi = \operatorname{curl}\mathbf{u}$ or $\operatorname{curl}\mathbf{w}$, φ satisfies
$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial\varphi}{\partial r} - \mathrm{i}\,\mathsf{k}_s\,\varphi\right) = 0\,, \tag{6.30}$$
uniformly in all directions.

2. With matrix P defined in (6.1), and for φ_P, φ_B defined as $\begin{pmatrix}\varphi_P\\ \varphi_B\end{pmatrix} = P^{-1}\begin{pmatrix}\nabla\cdot\mathbf{u}\\ \nabla\cdot\mathbf{w}\end{pmatrix}$, then φ_P and φ_B satisfies the outgoing Sommerfeld radiation condition with wavenumber k_P and k_B respectively:
$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial\varphi_P}{\partial r} - \mathrm{i}\,\mathsf{k}_P\,\varphi_P\right) = 0\,,\quad\text{and}\quad \lim_{r\to\infty}\sqrt{r}\left(\frac{\partial\varphi_B}{\partial r} - \mathrm{i}\,\mathsf{k}_B\,\varphi_B\right) = 0\,, \tag{6.31}$$
uniformly in all directions.

Remark 9. A similar definition can be proposed to define an outgoing solution if we work with the displacement (u,w), instead of the velocity $\mathbf{u}$ and $\mathbf{w}$. △

7 Generic solution to homogeneous equation on bounded domain

We consider the homogeneous poroelastic equations (3.30) on disc $\mathbb{B}_{(0,\mathfrak{a})}$. The solutions $(\mathbf{u},\mathbf{w},\boldsymbol{\tau},\mathrm{p})$ are given by equations (6.12), (6.13) and (6.14), while the potentials are given by(6.15). Hence, in a bounded domain, the potentials satisfy equation (6.16):

$$\begin{aligned}
\chi_{\mathrm{P}}(\boldsymbol{x}) &= \sum_{k\in\mathbb{Z}} a_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\,,\\
\chi_{\mathrm{B}}(\boldsymbol{x}) &= \sum_{k\in\mathbb{Z}} b_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\,,\\
\chi_{\mathrm{S}}(\boldsymbol{x}) &= \sum_{k\in\mathbb{Z}} c_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\,.
\end{aligned}$$

The series coefficients a_k, b_k, c_k are then determined by the boundary conditions imposed on $\partial\mathbb{B}_{(0,\mathfrak{a})}$, which are one of the four types listed in section 3.5. Here we only detail the solutions for type 1 and 3 (equations (3.31) and (3.33)).

7.1 Boundary conditions of type 1

We consider the poroelastic equations (3.30) on the disc $\mathbb{B}_{(0,\mathfrak{a})}$, with boundary conditions:

$$\mathbf{w}\cdot\mathbf{n} = g \quad,\quad \text{on } \partial\mathbb{B}_{(0,\mathfrak{a})}\,, \tag{7.1}$$
$$\boldsymbol{\tau}\cdot\mathbf{n} = \boldsymbol{h} \quad,\quad \text{on } \partial\mathbb{B}_{(0,\mathfrak{a})}\,. \tag{7.2}$$

In polar coordinates, $\boldsymbol{n} = \mathbf{e}_r$. Hence, $\mathbf{w}\cdot\boldsymbol{n} = \mathbf{w}_r$, $\boldsymbol{\tau}\cdot\boldsymbol{n} = \boldsymbol{\tau}_{rr}\,\mathbf{e}_r + \boldsymbol{\tau}_{r\theta}\,\mathbf{e}_\theta$. The boundary conditions are written as:

$$\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{w}_r = \mathfrak{s}\,\mathrm{i}\,\omega\,g\,,\quad \omega^2\,\boldsymbol{\tau}_{rr} = \omega^2\,\boldsymbol{h}_r\,,\quad \omega^2\,\boldsymbol{\tau}_{r\theta} = \omega^2\,\boldsymbol{h}_\theta \quad,\quad \partial\mathbb{B}_{(0,\mathfrak{a})}\,. \tag{7.3}$$

Next we expand the coefficient of each component in Fourier series. For the right hand-side,

$$g = \sum_{k\in\mathbb{Z}} g_k\,e^{\mathrm{i}\,k\,\theta}\,,\quad \boldsymbol{h}_r = \sum_{k\in\mathbb{Z}} \boldsymbol{h}_{r,k}\,e^{\mathrm{i}\,k\,\theta}\,,\quad \boldsymbol{h}_\theta = \sum_{k\in\mathbb{Z}} \boldsymbol{h}_{\theta,k}\,e^{\mathrm{i}\,k\,\theta}\,.$$

For the unknowns:

$$\mathbf{w}_r = \sum_{k\in\mathbb{Z}} \mathbf{w}_{r,k}\,e^{\mathrm{i}\,k\,\theta}\,,\quad \boldsymbol{\tau}_{rr} = \sum_{k\in\mathbb{Z}} \boldsymbol{\tau}_{rr,k}\,e^{\mathrm{i}\,k\,\theta}\,,\quad \boldsymbol{\tau}_{r\theta} = \sum_{k\in\mathbb{Z}} \boldsymbol{\tau}_{r\theta,k}\,e^{\mathrm{i}\,k\,\theta}\,.$$

Using (6.20), (6.22) and (6.23), we have:

$$
\begin{aligned}
\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{w}_{r,k} &= a_k\,\frac{\beta_{\mathrm{P}}}{\mathsf{s}_{\mathrm{P}}}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta} \;+\; b_k\,\frac{\beta_{\mathrm{B}}}{\mathsf{s}_{\mathrm{B}}}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta} \;+\; c_k\,\frac{\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}}}{\det A}\,\frac{\mathrm{i}k}{|\boldsymbol{x}|}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\,,\\
\omega^2\boldsymbol{\tau}_{rr,k} &= -\,\frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|}\,a_k\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \;+\; \frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{P}}^2\,|\boldsymbol{x}|^2}\,a_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \;+2\,\mu_{\mathrm{fr}}\,a_k\,\omega^2\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\\
&\quad -\,\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{P}}^2\,|\boldsymbol{x}|^2}\,a_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} - \frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|}\,b_k\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \;+\; \frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{B}}^2\,|\boldsymbol{x}|^2}\,b_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\\
&\quad +2\,\mu_{\mathrm{fr}}\,b_k\,\omega^2\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \;-\; \frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{B}}^2\,|\boldsymbol{x}|^2}\,b_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \;+\; \frac{2\,\mu_{\mathrm{fr}}}{\mathsf{s}_{\mathrm{S}}^2\,|\boldsymbol{x}|}\;c_k\,\omega\,\mathsf{s}_{\mathrm{S}}\,\mathrm{i}k\,\mathrm{J}_k^{'}(\,\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\\
&\quad +\,\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} \,+\, \mathrm{k}_{\mathrm{fr}} \,+\, M\alpha^2 \,+\, \alpha\,M\beta_{\mathrm{P}}\right)\;\; a_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\\
&\quad +\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} \,+\, \mathrm{k}_{\mathrm{fr}} \,+\, M\alpha^2 \,+\, \alpha\,M\beta_{\mathrm{B}}\right)\,b_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\,,\\
\omega^2\,\boldsymbol{\tau}_{r\theta,k} &= -\,\frac{2\,\mu_{\mathrm{fr}}\omega\,\mathrm{i}\,k}{|\boldsymbol{x}|\,\mathsf{s}_{\mathrm{P}}}\,a_k\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} + \frac{2\,\mathrm{i}\,\mu_{\mathrm{fr}}\,k}{|\boldsymbol{x}|^2\,\mathsf{s}_{\mathrm{P}}^2}\,a_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} \,-\, \frac{2\mu_{\mathrm{fr}}\omega\,\mathrm{i}\,k}{|\boldsymbol{x}|\,\mathsf{s}_{\mathrm{B}}}\,b_k\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)e^{\mathrm{i}\,k\,\theta}\\
&\quad +\,\frac{2\,\mathrm{i}\,\mu_{\mathrm{fr}}\,k}{|\boldsymbol{x}|^2\,\mathsf{s}_{\mathrm{B}}^2}\,b_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \;-\; \frac{\mu_{\mathrm{fr}}\,k^2}{|\boldsymbol{x}|^2\,\mathsf{s}_{\mathrm{S}}^2}\,c_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \,+\, \frac{\mu_{\mathrm{fr}}\,\omega}{|\boldsymbol{x}|\,\mathsf{s}_{\mathrm{S}}}\,c_k\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\\
&\quad -\,\mu_{\mathrm{fr}}\,\frac{\omega}{\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|}\,c_k\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \;+\mu_{\mathrm{fr}}\,\frac{k}{\mathsf{s}_{\mathrm{S}}^2\,|\boldsymbol{x}|^2}c_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\\
&\quad +\,\omega^2\,c_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \;-\; \mu_{\mathrm{fr}}\,\frac{k^2}{\mathsf{s}_{\mathrm{S}}^2\,|\boldsymbol{x}|^2}\,c_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}.
\end{aligned}
\tag{7.4}
$$

Imposing (7.3), we obtain a linear system satisfied by a_k, b_k, c_k in each mode k.

$$
\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}\begin{pmatrix} a_k\\ b_k\\ c_k\end{pmatrix} = \begin{pmatrix}\mathfrak{s}\,\mathrm{i}\,\omega\,g_k\\ \omega^2\,\boldsymbol{h}_{r,k}\\ \omega^2\,\boldsymbol{h}_{\theta,k}\end{pmatrix}\,, \tag{7.5}
$$

where the coefficient matrix is defined as:

$$
\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}} = \begin{pmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix} \tag{7.6}
$$

with

$$
\begin{aligned}
A_{11} &= \frac{\beta_{\mathrm{P}}}{\mathsf{s}_{\mathrm{P}}}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a})\,, \qquad A_{12} = \frac{\beta_{\mathrm{B}}}{\mathsf{s}_{\mathrm{B}}}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\mathfrak{a})\,, \qquad A_{13} = \frac{\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}}}{\det A}\,\frac{\mathrm{i}k}{\mathfrak{a}}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\mathfrak{a})\,,\\
A_{21} &= -\,\frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{P}}\,\mathfrak{a}}\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a}) \;+\; \frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{P}}^2\,\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a}) \;+\,2\,\mu_{\mathrm{fr}}\,\omega^2\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\\
&\qquad -\,\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{P}}^2\,\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a}) \;+\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} \,+\, \mathrm{k}_{\mathrm{fr}} \,+\, M\alpha^2 \,+\, \alpha\,M\beta_{\mathrm{P}}\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a})\,,\\
A_{22} &= -\,\frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{B}}\,\mathfrak{a}}\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a}) \;+\; \frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{B}}^2\,\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a}) \;+\,2\,\mu_{\mathrm{fr}}\,\omega^2\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\;e^{\mathrm{i}\,k\,\theta}\\
&\quad -\,\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{B}}^2\,\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\;e^{\mathrm{i}\,k\,\theta} +\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} \,+\, \mathrm{k}_{\mathrm{fr}} \,+\, M\alpha^2 \,+\, \alpha\,M\beta_{\mathrm{B}}\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\mathfrak{a})\,,
\end{aligned}
$$

and

$$A_{23} = \frac{2\mu_{\mathrm{fr}}}{\mathsf{s}_{\mathrm{S}}\mathfrak{a}}\,\omega\,\mathrm{i}k\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\mathfrak{a})\,, \qquad A_{31} = -\frac{2\omega\,\mu_{\mathrm{fr}}\mathrm{i}k}{\mathfrak{a}\mathsf{s}_{\mathrm{P}}}\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a}) + \frac{2\,\mu_{\mathrm{fr}}\mathrm{i}k}{\mathfrak{a}^2\mathsf{s}_{\mathrm{P}}^2}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a})\,,$$

$$A_{32} = -\frac{2\omega\,\mu_{\mathrm{fr}}\mathrm{i}k}{\mathfrak{a}\mathsf{s}_{\mathrm{B}}}\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{B}}\mathfrak{a}) + \frac{2\mu_{\mathrm{fr}}\mathrm{i}k}{\mathfrak{a}^2\mathsf{s}_{\mathrm{B}}^2}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\mathfrak{a})\,,$$

$$\begin{aligned} A_{33} = &-\frac{k^2\mu_{\mathrm{fr}}}{\mathfrak{a}^2\mathsf{s}_{\mathrm{S}}^2}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a}) + \frac{\omega\,\mu_{\mathrm{fr}}}{\mathfrak{a}\mathsf{s}_{\mathrm{S}}}\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a}) - \frac{\omega}{\mathsf{s}_{\mathrm{S}}\,\mathfrak{a}}\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a}) + \frac{k}{\mathsf{s}_{\mathrm{S}}^2\,\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a}) \\ &+ \omega^2\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a}) - \frac{k^2}{\mathsf{s}_{\mathrm{S}}^2\,\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\;. \end{aligned}$$

We define the eigenvalues as following:

Definition 3. The pulsation ω is a Type 1 eigenvalue if the system of poroelastic equations (3.30) associated with the boundary conditions

$$\begin{aligned} \mathbf{w}\cdot\mathbf{n} &= 0 \quad , \quad \text{on } \partial\mathbb{B}_{(0,\mathfrak{a})}\,, \\ \boldsymbol{\tau}\cdot\mathbf{n} &= 0 \quad , \quad \text{on } \partial\mathbb{B}_{(0,\mathfrak{a})}\,, \end{aligned} \tag{7.7}$$

admits a solution $(\mathbf{w},\,\boldsymbol{\tau})$ such that $\mathbf{w}\neq 0$, $\boldsymbol{\tau}\neq 0$. This also means that $\det \mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}(\omega) = 0$, where $\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}$ is the coefficients matrix defined in equation (7.6).

7.2 Boundary conditions of type 3

We consider the poroelastic equation on the disc $\mathbb{B}_{(0,\mathfrak{a})}$, with boundary conditions:

$$\mathbf{u} = \boldsymbol{h}\,, \qquad \mathrm{p} = g \quad , \quad \partial\mathbb{B}_{(0,\mathfrak{a})}\,. \tag{7.8}$$

We work in polar coordinates, $\boldsymbol{h} = \boldsymbol{h}_r\mathbf{e}_r + \boldsymbol{h}_\theta\,\mathbf{e}_\theta$ and $\mathbf{u} = \mathbf{u}_r\mathbf{e}_r + \mathbf{u}_\theta\,\mathbf{e}_\theta$. The boundary conditions are written as:

$$\mathfrak{s}\,\mathrm{i}\omega\,\mathbf{u}_r = \mathfrak{s}\,\mathrm{i}\omega\,\boldsymbol{h}_r\,, \quad \mathfrak{s}\,\mathrm{i}\omega\,\mathbf{u}_\theta = \mathfrak{s}\,\mathrm{i}\omega\,\boldsymbol{h}_\theta\,, \quad \mathrm{p} = g \quad , \quad \partial\mathbb{B}_{(0,\mathfrak{a})}\,. \tag{7.9}$$

Next we expand the coefficient of each component in Fourier series. For the right hand-side,

$$\boldsymbol{h}_r = \sum_{k\in\mathbb{Z}} \boldsymbol{h}_{r,k}\,e^{\mathrm{i}\,k\,\theta}\,, \quad \boldsymbol{h}_\theta = \sum_{k\in\mathbb{Z}} \boldsymbol{h}_{\theta,k}\,e^{\mathrm{i}\,k\,\theta}\,, \quad g = \sum_{k\in\mathbb{Z}} g_k\,e^{\mathrm{i}\,k\,\theta}\,.$$

For the unknowns:

$$\mathbf{u}_r = \sum_{k\in\mathbb{Z}} \mathbf{u}_{r,k}\,e^{\mathrm{i}\,k\,\theta}\,, \quad \mathbf{u}_\theta = \sum_{k\in\mathbb{Z}} \mathbf{u}_{\theta,k}\,e^{\mathrm{i}\,k\,\theta}\,, \quad \mathrm{p} = \sum_{k\in\mathbb{Z}} \mathrm{p}_k\,e^{\mathrm{i}\,k\,\theta}\,.$$

Using (6.19) and (6.24), we have:

$$\begin{aligned} \mathfrak{s}\,\mathrm{i}\omega\,\mathbf{u}_{r,k} &= \sum_{k\in\mathbb{Z}} a_k\,\mathsf{s}_{\mathrm{P}}^{-1}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta} + \sum_{k\in\mathbb{Z}} b_k\,\mathsf{s}_{\mathrm{B}}^{-1}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta} - \sum_{k\in\mathbb{Z}} c_k\,\mathsf{s}_{\mathrm{S}}^{-2}\,\frac{\mathrm{i}k}{|\boldsymbol{x}|}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\,, \\ \mathfrak{s}\,\mathrm{i}\omega\,\mathbf{u}_{\theta,k} &= \sum_{k\in\mathbb{Z}} a_k\,\mathsf{s}_{\mathrm{P}}^{-2}\,\frac{\mathrm{i}k}{|\boldsymbol{x}|}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta} + \sum_{k\in\mathbb{Z}} b_k\,\mathsf{s}_{\mathrm{B}}^{-2}\,\frac{\mathrm{i}k}{|\boldsymbol{x}|}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta} + \sum_{k\in\mathbb{Z}} c_k\,\mathsf{s}_{\mathrm{S}}^{-1}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\,, \\ \mathrm{p}_k &= -\sum_{k\in\mathbb{Z}} a_k\,M\left(\beta_{\mathrm{P}} + \alpha\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}k\theta} - \sum_{k\in\mathbb{Z}} b_k\,M\left(\beta_{\mathrm{B}} + \alpha\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}k\theta}\,. \end{aligned} \tag{7.10}$$

Imposing (7.9), we obtain a linear system satisfied by a_k, b_k, c_k in each mode k:

$$\mathbb{A}_k^{\mathbf{u},\mathrm{p}}\begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} = \begin{pmatrix} \mathfrak{s}\,\mathrm{i}\,\omega\,\boldsymbol{h}_{r,k} \\ \mathfrak{s}\,\mathrm{i}\,\omega\,\boldsymbol{h}_{\theta,k} \\ g_k \end{pmatrix}\,, \tag{7.11}$$

where the coefficients matrix $\mathbb{A}_k^{\mathbf{u},\mathrm{p}}$ is defined as:

$$\mathbb{A}_k^{\mathbf{u},\mathrm{p}} = \begin{pmatrix} \mathsf{s}_\mathrm{P}^{-1}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_\mathrm{P}\,\mathfrak{a}) & \mathsf{s}_\mathrm{B}^{-1}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_\mathrm{S}\,\mathfrak{a}) & -\,\mathsf{s}_\mathrm{S}^{-2}\,\dfrac{\mathrm{i}k}{\mathfrak{a}}\mathrm{J}_k(\omega\,\mathsf{s}_\mathrm{S}\,\mathfrak{a}) \\ \mathsf{s}_\mathrm{P}^{-2}\,\dfrac{\mathrm{i}k}{\mathfrak{a}}\mathrm{J}_k(\omega\,\mathsf{s}_\mathrm{P}\,\mathfrak{a}) & \mathsf{s}_\mathrm{B}^{-2}\,\dfrac{\mathrm{i}k}{\mathfrak{a}}\mathrm{J}_k(\omega\,\mathsf{s}_\mathrm{B}\,\mathfrak{a}) & \mathsf{s}_\mathrm{S}^{-1}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_\mathrm{S}\,\mathfrak{a}) \\ -M\left(\upbeta_\mathrm{P}\ +\ \alpha\right)\mathrm{J}_k(\omega\,\mathsf{s}_\mathrm{P}\,\mathfrak{a}) & -\,M\left(\upbeta_\mathrm{B}\ +\ \alpha\right)\mathrm{J}_k(\omega\,\mathsf{s}_\mathrm{B}\,\mathfrak{a}) & 0 \end{pmatrix}. \tag{7.12}$$

Definition 4. The pulsation ω is a Type 3 eigenvalue if the system of poroelastic equations (3.30) associated with the boundary conditions

$$\begin{aligned} \mathbf{u} \;&=\; 0 \quad , \quad \text{on } \partial\mathbb{B}_{(0,\mathfrak{a})}\,, \\ \mathrm{p} \;&=\; 0 \quad , \quad \text{on } \partial\mathbb{B}_{(0,\mathfrak{a})}\,, \end{aligned} \tag{7.13}$$

admits a solution $(\mathbf{u}, \mathrm{p})$ such that $\mathbf{u} \neq 0$, $\mathrm{p} \neq 0$. This also means that $\det \mathbb{A}_k^{\mathbf{u},\mathrm{p}}(\omega) = 0$, where $\mathbb{A}_k^{\mathbf{u},\mathrm{p}}$ is the coefficients matrix defined in equation (7.11).

7.3 Numerical tests on bounded domain

We will investigate the stability of the coefficient matrices for the first few modes $k = 0, ..., 5$, by looking at the absolute value of the determinant of the coefficients matrix. The objective is to determine if we can find generalized eigenvalues where the determinant vanishes. We test with sandstone, and vary the value of viscosity of this material, *cf.* table 1.

For all tests, the cross section radius is $\mathfrak{a} = 1$m. Recall that we can use four types of boundary conditions (see equations (3.31), (3.32), (3.33) and (3.34)). Here we only test the boundary conditions of type 1 and 3, which means we only study the determinant of $\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}$ (7.6) and $\mathbb{A}^{\mathbf{u},\mathrm{p}}k$ (7.12), as a function of the frequency f. We consider a frequency range between 1Hz and 10MHz to compare with the results in [1] and [16]. Note that the interval ($\omega\mathfrak{a} \leq 1500\text{m.s}^{-1}$) in our plot is more relevant to geophysical experiments[3]. To determine the nature of the peaks, we refine around the peaks in a procedure described in algorithm 1. This will be used in all the remaining tests of the report.

Start Suppose x_center is the local minimum on the interval $[a, b]$, with h the current stepsize of the sequence. Say $x_\text{center} = N\,h$, and $a = x_\text{center} - m\,h$, $b = x_\text{center} + m'\,h$, with m, $m' \geq 5$, hence at least 5 points before and after x_center in the sequence. The interval in consideration always has to satisfy criteria ($\star$), which requires that the function decreases for $x \leq x_\text{center}$ and increases for $x \geq x_\text{center}$.

Update A new interval $[a, b]$ is chosen, centered around the previous miminum, and thus satisfies the criteria ($\star$). We plot the value of the function on this new interval, with the new step on the frequency equals to $\frac{h}{10}$. In this way, we are on a smaller interval with a finer grid.

Iteration A new center x_center is now the new local minimum of the function on this interval. We go back to the start.

Stop criteria The loop is stopped if the size of h is lower than the machine precision or if the minimum value on the interval at the current iteration does not differ from that of the previous one by ε.

Algorithm 1: Algorithm for detecting the modes of inversibility. There are two behaviours when we refine around a peak: In the case where it is a true zero, the value of the function shown on log scale will decrease until the machine precision on the h interval. If it is not a true zero, the values of the absolute determinant will stabilize to a fixed lower bound. In fact, the ε is implemented qualitatively, i.e. by the observation of the curve. In particular, in the first iterations, we observe a curved down bump, but after a few iterations, we only obtain an horizontal line, which means that the value on the zoommed interval stopped descending.

The results are reported in the following figures:

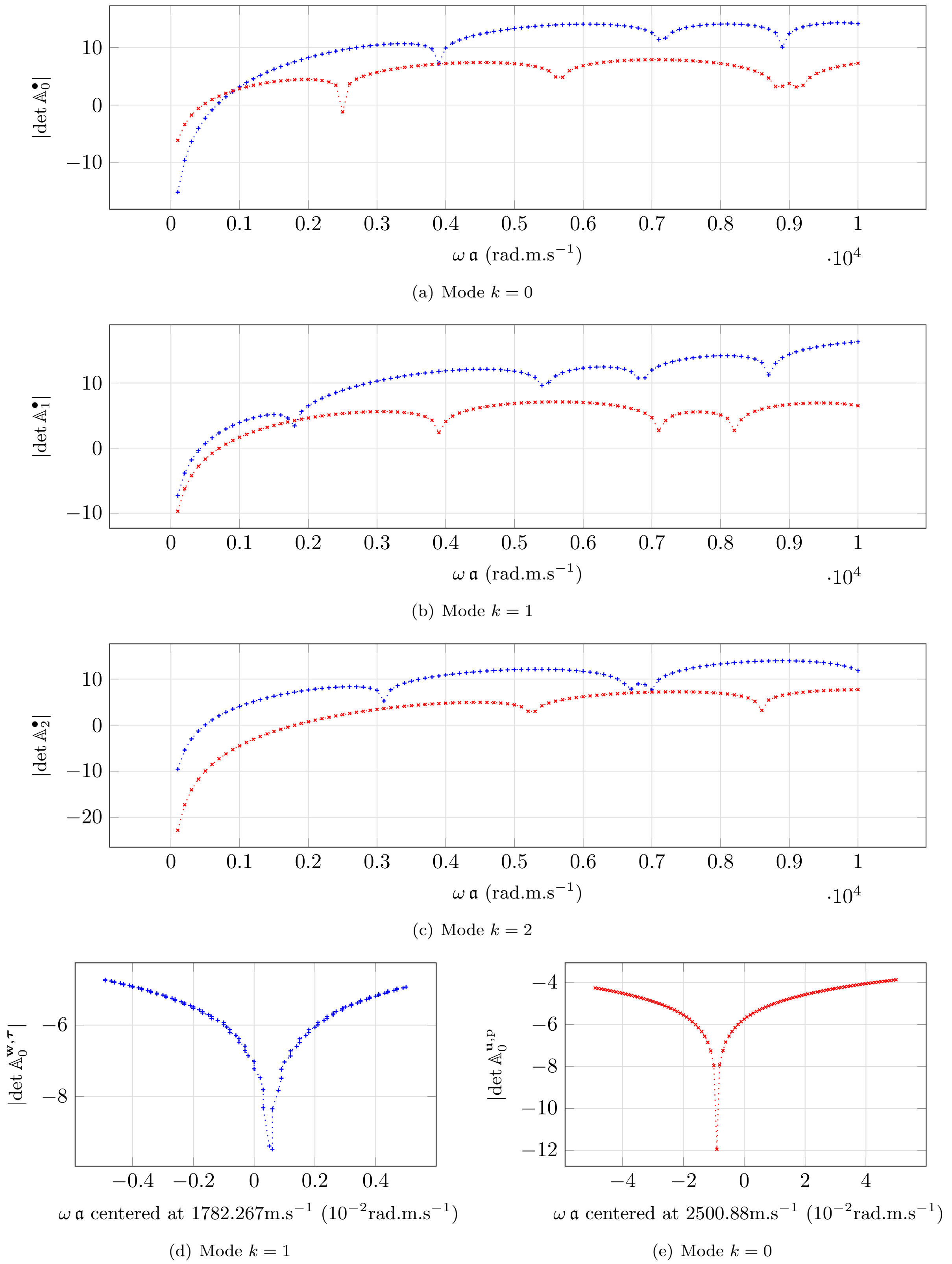
[3] In [15], and [22], frequencies up to 600Hz are used, on a domain of interest of 10^2m.

- Sandstone with no viscosity figures 1 and 2.
- Sandstone with viscosity figures 3 and 4.
- Sand 1 with no viscosity, varying the value of μ_{fr} in figure 5.

Observations From these experiments, we obtain the following observations.

- In the geophysical range ($\omega\mathfrak{a} \leq 1500\text{m.s}^{-1}$), generalized are present eigenvalues for non-viscous problems.
- On the initial interval $[0, 10^4]$ m.s^{-1}, the curves for material of sandstone with viscosity ressemble those of with no viscosity. The curves represent isolated peaks, however, the case with viscosity presents less peaks for the same range of frequency. After the zoom procedure, their behaviours are different. We applied the zoom procedure to each of the peaks in the graph of absolute determinant of $\mathbb{A}_0^{\mathbf{w},\boldsymbol{\tau}}$ and $\mathbb{A}_0^{\mathbf{u},\text{p}}$. We note that there are differences between those with and without viscosity. There exist generalized eigenvalues for the first case manifested by the sharp peaks for both boundary conditions in figures 1 and 2.

  Before zooming, the value in the neighbourhood of a peak in consideration is around 10^{-2}. However, after several zooms, the value in this refined neighbourhood drastically drops to 10^{-10} for $\mathbb{A}_0^{\mathbf{w},\boldsymbol{\tau}}$ and 10^{-16} for $\mathbb{A}_0^{\mathbf{u},\text{p}}$. We only show few examples of this refinement. With finer refinement, the value of the refined neighbourhood will drop to the machine precision. On the other hand, with viscosity, we do not have this behaviour. Although there are apparent peaks before zoom in figures 3 and 4, however, when zoomed around the sharpest peak, the value of the determinant on the refined neighbourhoods stays bounded below, and the sharp peaks become smooth concave up curves. Hence, there are no generalized eigenvalue in this case.
- In the case of sandstone, we have similar pattern of peaks for both boundary conditions, however, the generalized eigenvalues are not the same.
- Note that the curves here present small peaks, compared to the absence of generalized eigenvalues in the case in figure 8, in which the curves are completely free of peaks.
- The frame shear modulus has an influence on the peaks. We observe in figure 5 that when the value of the shear frame modulus increases, the number of the peaks, hence of eigenvalues decreases.



(a) Mode $k = 0$

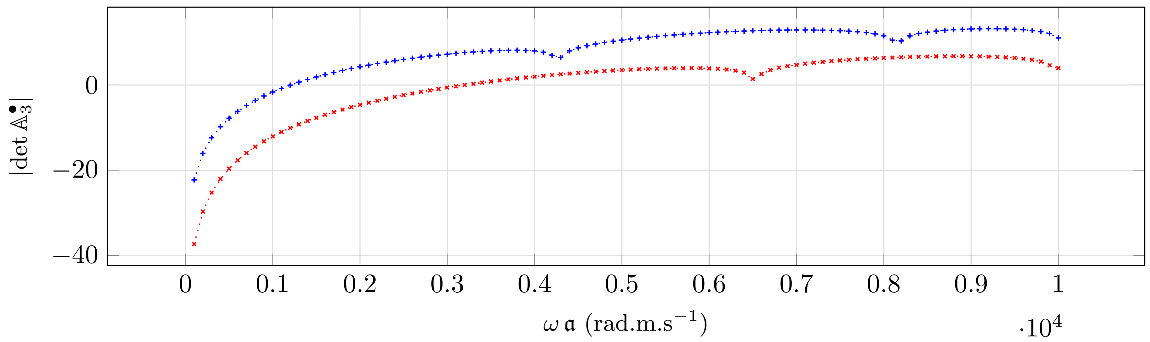
(b) Mode $k = 1$

(c) Mode $k = 2$

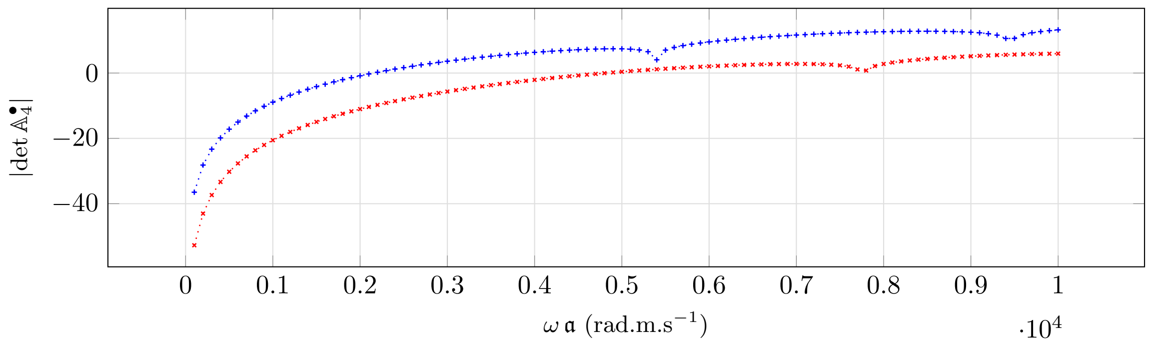
(d) Mode $k = 1$

(e) Mode $k = 0$

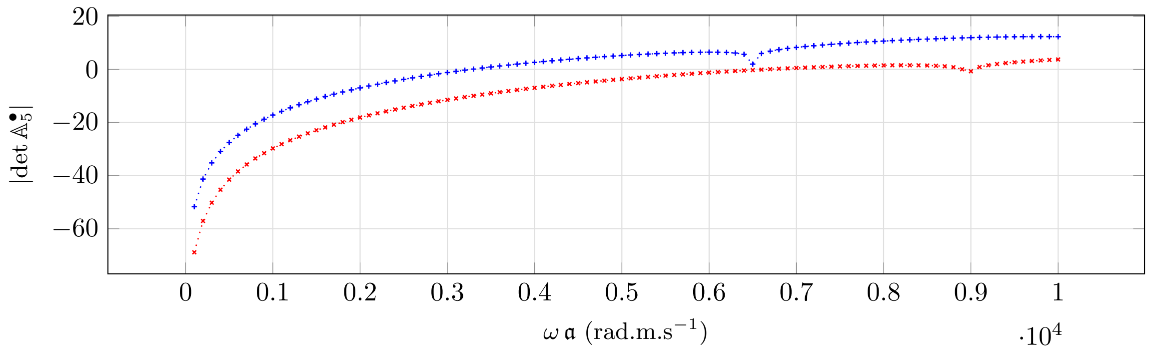
Figure 1: Determinant of the coefficients matrix (log scale) in a bounded domain for k in $0:2$ for sandstone with no viscosity. The matrices corresponding with types of boundary conditions 1 and 3 are considered: $\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}$ (7.6) in blue and $\mathbb{A}_k^{\mathbf{u},\mathrm{p}}$ (7.12) in red .



(a) Mode $k = 3$

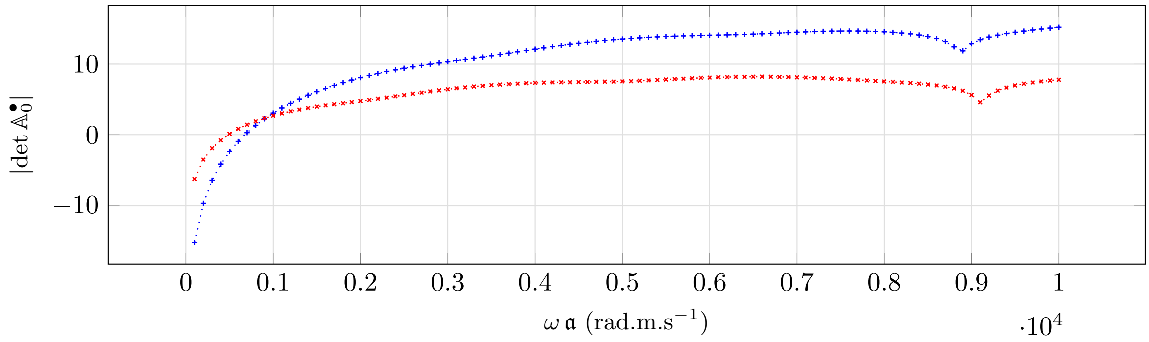


(b) Mode $k = 4$

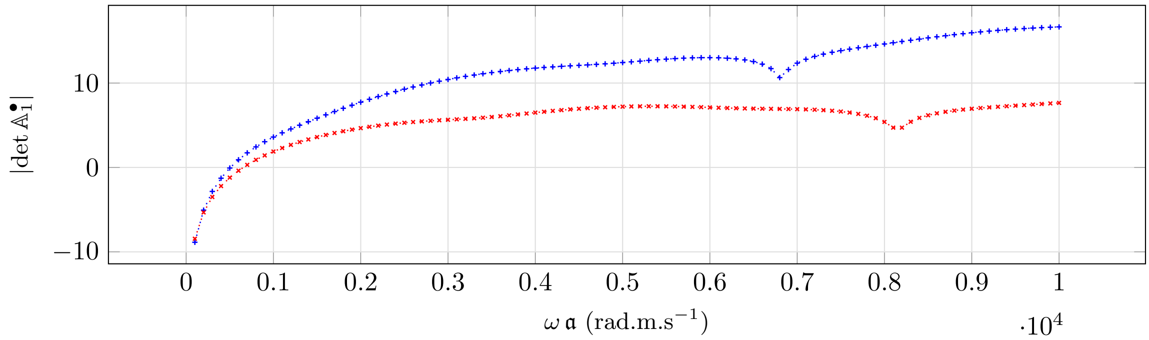


(c) Mode $k = 5$

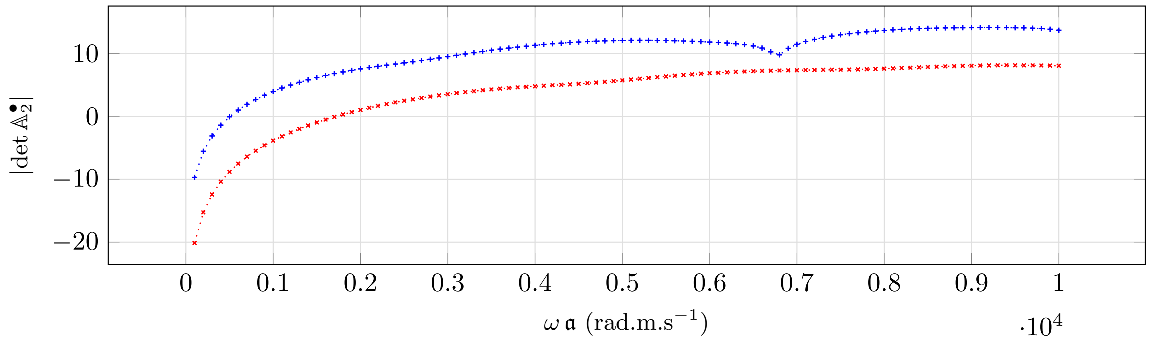
Figure 2: Determinant of the coefficients matrix (log scale) in a bounded domain for k in $3:5$ for sandstone with no viscosity. The matrices corresponding with types of boundary conditions 1 and 3 are considered: $\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}$ (7.6)in blue and $\mathbb{A}_k^{\mathbf{u},\mathrm{p}}$ (7.12) in red .



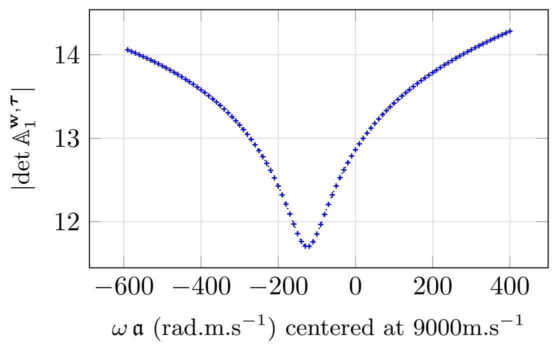
(a) Mode $k = 0$



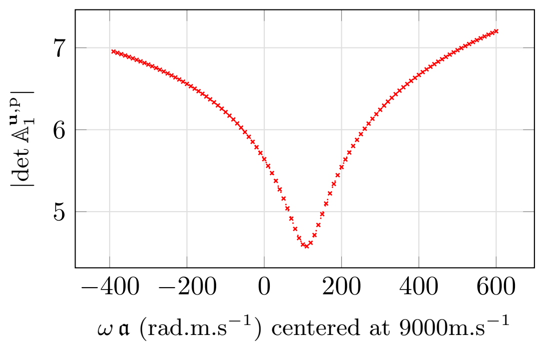
(b) Mode $k = 1$



(c) Mode $k = 2$

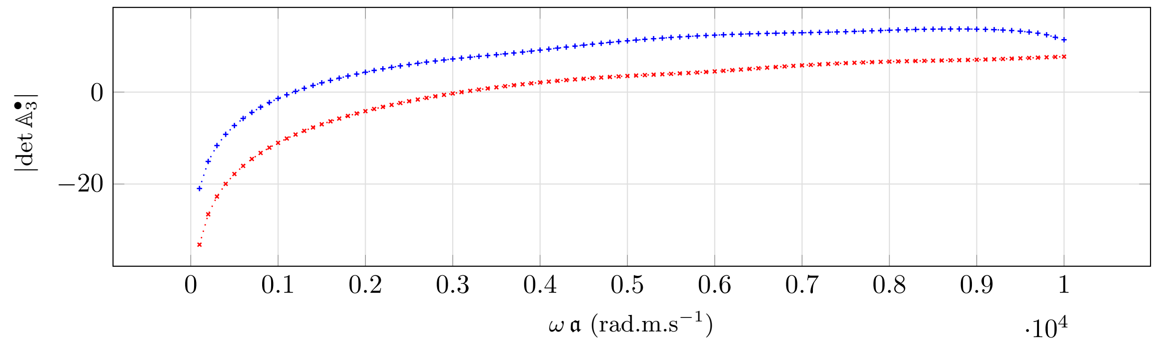


(d) Mode $k = 0$

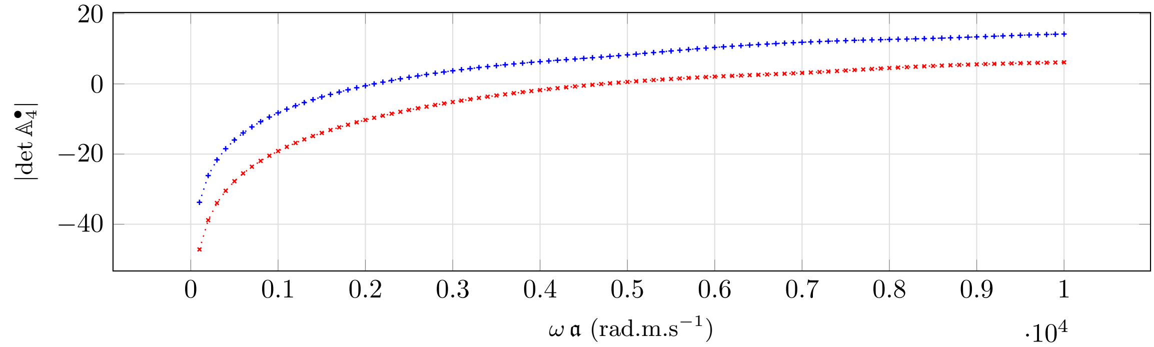


(e) Mode $k = 0$

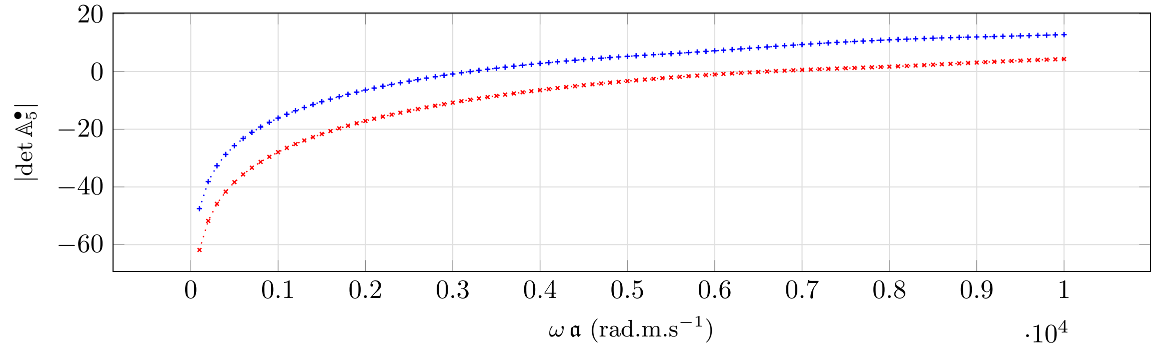
Figure 3: Determinant of the coefficients matrix (log scale) in a bounded domain for k in $0:2$ for a sandstone medium with viscosity $\eta \neq 0$. The matrices corresponding with types of boundary conditions 1 and 3 are considered: $\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}$ (7.6) in blue ⋯+⋯ and $\mathbb{A}_k^{\mathbf{u},\mathrm{p}}$ (7.12) in red ⋯×⋯.



(a) Mode $k = 3$



(b) Mode $k = 4$



(c) Mode $k = 5$

Figure 4: Determinant of the coefficients matrix (log scale) in a bounded domain for k in $3:5$ for a sandstone medium with viscosity $\eta \neq 0$. The matrices corresponding with types of boundary conditions 1 and 3 are considered: $\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}$ (7.6) in blue and $\mathbb{A}_k^{\mathbf{u},\mathrm{p}}$ (7.12) in red.

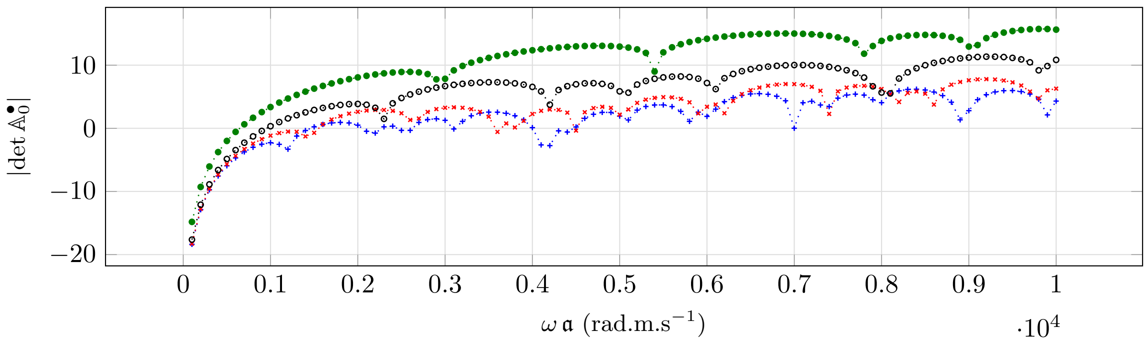


Figure 5: Comparison of the determinant of the coefficients matrix (log scale) in a bounded domain for $k = 0$ for different values of $\mu_{\rm fr}$ for a medium composed of sand1 with no viscosity. The matrix corresponding with types of boundary conditions 1 is considered: $\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}$. represents the case $\mu_{\rm fr} = 0.5$ GPa, represents the case $\mu_{\rm fr} = 1$ GPa, $\mu_{\rm fr} = 5$ GPa and $\mu_{\rm fr} = 50$ GPa, .

8 Scattering of a plane wave by an impenetrable medium

Consider the scattering of a time-harmonic plane wave by an impenetrable infinite cylinder (see figure 6). The total wave is a superposition of the incident plane wave and the reflected wave with each quantity satisfying poroelastic equations (3.30) in $\mathbb{R}^2 \setminus \mathbb{B}_{(0,\mathfrak{a})}$, also listed below in (8.1) and (8.6), according to the type of boundary conditions. The solutions $(\mathbf{u}, \mathbf{w}, \boldsymbol{\tau}, \mathrm{p})$ are given by equations (6.12), (6.13) and (6.14), while the potentials are given by (6.15). The unknown is the reflected wave which is outgoing, this means that it satisfies the Sommerfeld radiation condition(6.29), and is in addition uniquely determined by how the obstacle scatters the plane wave. Hence, the potentials corresponding to the reflected wave are given in equation (6.17):

$$
\begin{aligned}
\chi_{\mathrm{P}}(\boldsymbol{x}) &= \sum_{k\in\mathbb{Z}} a_k \, \mathrm{H}_k^{(1)}(\omega \, \mathsf{s}_{\mathrm{P}} \, |\boldsymbol{x}|) \; e^{\mathrm{i}\,k\,\theta} \,, \\
\chi_{\mathrm{B}}(\boldsymbol{x}) &= \sum_{k\in\mathbb{Z}} b_k \, \mathrm{H}_k^{(1)}(\omega \, \mathsf{s}_{\mathrm{B}} \, |\boldsymbol{x}|) \; e^{\mathrm{i}\,k\,\theta} \,, \\
\chi_{\mathrm{S}}(\boldsymbol{x}) &= \sum_{k\in\mathbb{Z}} c_k \, \mathrm{H}_k^{(1)}(\omega \, \mathsf{s}_{\mathrm{S}} \, |\boldsymbol{x}|) \; e^{\mathrm{i}\,k\,\theta} \,.
\end{aligned}
$$

The series coefficients a_k, b_k, c_k are then determined by the boundary conditions imposed on the interface Γ. We will consider the boundary conditions of type 1 in 8.1 and of type 3 in 8.2.

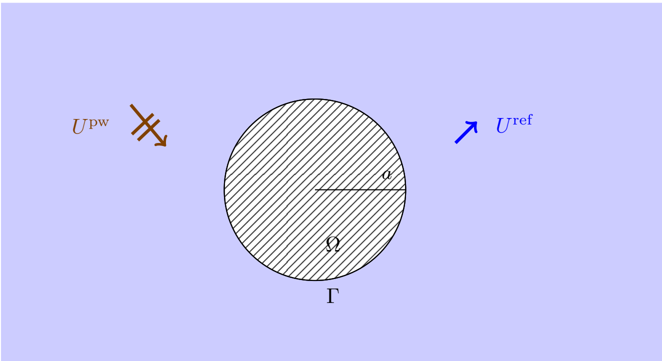


Figure 6: Scattering of a plane wave by an impenetrable solid inclusion. The inclusion occupies the domain denoted by Ω. The cross section of the inclusion is a disc of radius denoted by $\mathfrak{a}$. How the obstacle scatters the plane wave is mathematically described by boundary conditions, for example (3.31) or (3.33).

8.1 Boundary conditions of type 1

For $\bullet$ = total, ref, pw, we denote by

$$
U^{\bullet} = \begin{pmatrix} \mathbf{u}^{\bullet} \\ \mathbf{w}^{\bullet} \\ \boldsymbol{\tau}^{\bullet} \\ \mathrm{p}^{\bullet} \end{pmatrix}
$$

the total wave, the reflected wave and the incident plane wave correspondingly. The unknown reflected wave solves the poroelastic problem:

$$
\begin{cases}
U^{\mathrm{ref}} \text{ solves the poroelastic equations (3.30) in } \mathbb{R}^2 \setminus \Omega\,; \\[2ex]
U^{\mathrm{ref}} \text{ is outgoing by definition (6.29);} \\[2ex]
\text{Boundary conditions on the interface } \Gamma \\
\quad \mathbf{w}^{\mathrm{pw}} \cdot \mathbf{n} + \mathbf{w}^{\mathrm{ref}} \cdot \mathbf{n} = 0 \quad , \quad \text{on } \partial\mathbb{B}_{(0,\mathfrak{a})}\,; \\
\quad \boldsymbol{\tau}^{\mathrm{pw}} \cdot \mathbf{n} + \boldsymbol{\tau}^{\mathrm{ref}} \cdot \mathbf{n} = 0 \quad , \quad \text{on } \partial\mathbb{B}_{(0,\mathfrak{a})}\,.
\end{cases}
\tag{8.1}
$$

In the current geometry, $\boldsymbol{n} = \mathbf{e}_r$. Hence,

$$\mathbf{w}\cdot\boldsymbol{n} = \mathbf{w}_r\,, \qquad \boldsymbol{\tau}\cdot\boldsymbol{n} = \boldsymbol{\tau}_{rr}\,\mathbf{e}_r + \boldsymbol{\tau}_{r\theta}\,\mathbf{e}_\theta\,,$$
$$\mathbf{w}^{\mathrm{pw}}\cdot\boldsymbol{n} = \mathbf{w}_r^{\mathrm{pw}}\,, \qquad \boldsymbol{\tau}^{\mathrm{pw}}\cdot\boldsymbol{n} = \boldsymbol{\tau}_{rr}^{\mathrm{pw}}\,\mathbf{e}_r + \boldsymbol{\tau}_{r\theta}^{\mathrm{pw}}\,\mathbf{e}_\theta\,.$$

The boundary conditions are written as:

$$\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{w}_r \;=\; \mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{w}_r^{\mathrm{pw}}\,,\quad \omega^2\,\boldsymbol{\tau}_{rr} \;=\; \omega^2\,\boldsymbol{\tau}_{rr}^{\mathrm{pw}}\,,\quad \omega^2\,\boldsymbol{\tau}_{r\theta} \;=\; \omega^2\,\boldsymbol{\tau}_{r\theta}^{\mathrm{pw}}\quad,\quad \text{on } \partial\mathbb{B}_{(0,\mathfrak{a})}\,. \tag{8.2}$$

Next we expand the coefficients of each component in Fourier series. For the right hand-side,

$$\mathbf{w}_r^{\mathrm{pw}} = \sum_{k\in\mathbb{Z}} \mathbf{w}_{r,k}^{\mathrm{pw}}\,e^{\mathrm{i}\,k\,\theta}\,,\quad \boldsymbol{\tau}_{rr}^{\mathrm{pw}} = \sum_{k\in\mathbb{Z}} \boldsymbol{\tau}_{rr,k}^{\mathrm{pw}}\,e^{\mathrm{i}\,k\,\theta}\,,\quad ,\boldsymbol{\tau}_{r\theta}^{\mathrm{pw}} = \sum_{k\in\mathbb{Z}} \boldsymbol{\tau}_{r\theta,k}^{\mathrm{pw}}\,e^{\mathrm{i}\,k\,\theta}\,.$$

For the unknowns:

$$\mathbf{w}_r = \sum_{k\in\mathbb{Z}} \mathbf{w}_{r,k}\,e^{\mathrm{i}\,k\,\theta}\,,\quad \boldsymbol{\tau}_{rr} = \sum_{k\in\mathbb{Z}} \boldsymbol{\tau}_{rr,k}\,e^{\mathrm{i}\,k\,\theta}\,,\quad \boldsymbol{\tau}_{r\theta} = \sum_{k\in\mathbb{Z}} \boldsymbol{\tau}_{r\theta,k}\,e^{\mathrm{i}\,k\,\theta}\,.$$

Using (6.20), (6.22) and (6.23), we have:

$$\begin{aligned}
\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{w}_{r,k} =&\, a_k\,\frac{\beta_{\mathrm{P}}}{\mathsf{s}_{\mathrm{P}}}\,\omega\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}k\theta} \;+\; b_k\,\frac{\beta_{\mathrm{B}}}{\mathsf{s}_{\mathrm{B}}}\,\omega\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}k\theta} \;+\; c_k\,\frac{\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}}}{\det A}\,\frac{\mathrm{i}k}{|\boldsymbol{x}|}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}k\theta}\,,\\
\omega^2\boldsymbol{\tau}_{rr,k} =& -\frac{2\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|}\,a_k\,\mathrm{H}_{k+1}^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} \;+\frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{P}}^2\,|\boldsymbol{x}|^2}\,a_k\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} + 2\mu_{\mathrm{fr}}\,a_k\,\omega^2\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta}\\
&-\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{P}}^2\,|\boldsymbol{x}|^2}\,a_k\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} - \frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|}\,b_k\,\mathrm{H}_{k+1}^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} \;+\frac{2\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{B}}^2\,|\boldsymbol{x}|^2}b_k\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta}\\
&+2\mu_{\mathrm{fr}}\,b_k\,\omega^2\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} \;-\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{B}}^2\,|\boldsymbol{x}|^2}b_k\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} \;+\frac{2\,\mu_{\mathrm{fr}}}{\mathsf{s}_{\mathrm{S}}^2 r}c_k\,\omega\,\mathsf{s}_{\mathrm{S}}\,\mathrm{i}k\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta}\\
&+\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} \;+\; \mathrm{k}_{\mathrm{fr}} \;+\; M\alpha^2 \;+\; \alpha\,M\beta_{\mathrm{P}}\right)\;a_k\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\\
&+\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} \;+\; \mathrm{k}_{\mathrm{fr}} \;+\; M\alpha^2 \;+\; \alpha\,M\beta_{\mathrm{B}}\right)\,b_k\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\,,\\
\omega^2\,\boldsymbol{\tau}_{r\theta,k} =& -\frac{2\mu_{\mathrm{fr}}\omega\,\mathrm{i}k}{|\boldsymbol{x}|\mathsf{s}_{\mathrm{P}}}a_k\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{P}}|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} + \frac{2\mathrm{i}\,\mu_{\mathrm{fr}}k}{|\boldsymbol{x}|^2\,\mathsf{s}_{\mathrm{P}}^2}a_k\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} - \frac{2\mu_{\mathrm{fr}}\omega\,\mathrm{i}\,k}{|\boldsymbol{x}|\,\mathsf{s}_{\mathrm{B}}}b_k\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{B}}|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta}\\
&+\frac{2\,\mathrm{i}\,\mu_{\mathrm{fr}}\,k}{|\boldsymbol{x}|^2\mathsf{s}_{\mathrm{B}}^2}\,b_k\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \;-\frac{\mu_{\mathrm{fr}}\,k^2}{|\boldsymbol{x}|^2\,\mathsf{s}_{\mathrm{S}}^2}\,c_k\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \;+\frac{\mu_{\mathrm{fr}}\,\omega}{|\boldsymbol{x}|\,\mathsf{s}_{\mathrm{S}}}\,c_k\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\\
&-\mu_{\mathrm{fr}}\,\frac{\omega}{\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|}\,c_k\,\mathrm{H}_{k+1}^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \;+\mu_{\mathrm{fr}}\,\frac{k}{\mathsf{s}_{\mathrm{S}}^2\,|\boldsymbol{x}|^2}c_k\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}\\
&+\omega^2\,c_k\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta} \;-\mu_{\mathrm{fr}}\,\frac{k^2}{\mathsf{s}_{\mathrm{S}}^2\,|\boldsymbol{x}|^2}\,c_k\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k\,\theta}.
\end{aligned} \tag{8.3}$$

Imposing (8.2), we obtain a linear system satisfied by a_k, b_k, c_k in each mode k.

$$\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}\begin{pmatrix}a_k\\ b_k\\ c_k\end{pmatrix} = \begin{pmatrix}-\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{w}_r^{\mathrm{pw}}\\ -\omega^2\,\boldsymbol{\tau}_{rr}^{\mathrm{pw}}\\ -\omega^2\,\boldsymbol{\tau}_{r\theta}^{\mathrm{pw}}\end{pmatrix}\,, \tag{8.4}$$

where the coefficients matrix is defined as:

$$\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}} = \begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix} \tag{8.5}$$

with

$$A_{11} = \frac{\beta_{\mathrm{P}}}{\mathsf{s}_{\mathrm{P}}}\,\omega\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a})\,, \qquad A_{12} = \frac{\beta_{\mathrm{B}}}{\mathsf{s}_{\mathrm{B}}}\,\omega\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{S}}\mathfrak{a}) \qquad A_{13} = \frac{\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}}}{\det A}\,\frac{\mathrm{i}k}{\mathfrak{a}}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\mathfrak{a})\,,$$

$$\begin{aligned} A_{21} = &-\frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{P}}\,\mathfrak{a}}\,\mathrm{H}_{k+1}^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\;e^{\mathrm{i}\,k\,\theta} \;+\; \frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{P}}^2\,\mathfrak{a}^2}\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a}) \;+\; 2\,\mu_{\mathrm{fr}}\,\omega^2\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\\ &-\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{P}}^2\,\mathfrak{a}^2}\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a}) \;+\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} \;+\; \mathrm{k}_{\mathrm{fr}} \;+\; M\alpha^2 \;+\; \alpha\,M\beta_{\mathrm{P}}\right)\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a})\,, \end{aligned}$$

$$\begin{aligned} A_{22} = &-\frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{B}}\,\mathfrak{a}}\,\mathrm{H}_{k+1}^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a}) \;+\; \frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{B}}^2\,\mathfrak{a}^2}\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a}) \;+\; 2\,\mu_{\mathrm{fr}}\,\omega^2\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\\ &-\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{B}}^2\,\mathfrak{a}^2}\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a}) \;+\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} \;+\; \mathrm{k}_{\mathrm{fr}} \;+\; M\alpha^2 \;+\; \alpha\,M\beta_{\mathrm{B}}\right)\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\mathfrak{a})\,, \end{aligned}$$

and

$$A_{23} = \frac{2\mu_{\mathrm{fr}}}{\mathsf{s}_{\mathrm{S}}\mathfrak{a}}\,\omega\,\mathrm{i}k\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{S}}\mathfrak{a})\,,$$

$$A_{31} = -\frac{2\omega\,\mu_{\mathrm{fr}}\mathrm{i}k}{\mathfrak{a}\mathsf{s}_{\mathrm{P}}}\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a}) + \frac{2\,\mu_{\mathrm{fr}}\mathrm{i}k}{\mathfrak{a}^2\mathsf{s}_{\mathrm{P}}^2}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a})\,,$$

$$A_{32} = -\frac{2\omega\,\mu_{\mathrm{fr}}\mathrm{i}k}{\mathfrak{a}\mathsf{s}_{\mathrm{B}}}\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{B}}\mathfrak{a}) + \frac{2\mu_{\mathrm{fr}}\mathrm{i}k}{\mathfrak{a}^2\mathsf{s}_{\mathrm{B}}^2}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\mathfrak{a})\,,$$

$$\begin{aligned} A_{33} = &-\frac{k^2\mu_{\mathrm{fr}}}{\mathfrak{a}^2\mathsf{s}_{\mathrm{S}}^2}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a}) + \frac{\omega\,\mu_{\mathrm{fr}}}{\mathfrak{a}\mathsf{s}_{\mathrm{S}}}\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a}) \;-\; \frac{\omega}{\mathsf{s}_{\mathrm{S}}\,\mathfrak{a}}\,\mathrm{H}_{k+1}^{(1)}(\mathsf{s}_{\mathrm{S}}\,\mathfrak{a}) \;+\; \frac{k}{\mathsf{s}_{\mathrm{S}}^2\,\mathfrak{a}^2}\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\;,\\ &+\,\omega^2\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\;e^{\mathrm{i}\,k\,\theta} \;-\; \frac{k^2}{\mathsf{s}_{\mathrm{S}}^2\,\mathfrak{a}^2}\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\;. \end{aligned}$$

8.2 Boundary conditions of type 3

In this case, the unknown reflected wave solves the following poroelastic problem:

$$\begin{cases} U^{\mathrm{ref}} \text{ solves the poroelastic equations (3.30) in } \mathbb{R}^2 \setminus \Omega\,;\\[4pt] U^{\mathrm{ref}} \text{ is outgoing by definition (6.29) ;}\\[4pt] \text{Boundary conditions on the interface } \Gamma\\ \quad \mathbf{v}^{\mathrm{pw}} + \mathbf{v}^{\mathrm{ref}} = 0 \;\text{ on } \Gamma\,,\\ \quad \mathrm{p}^{\mathrm{pw}} + \mathrm{p}^{\mathrm{ref}} = 0 \;\text{ on } \Gamma\,. \end{cases} \tag{8.6}$$

We work in polar coordinates, $\mathbf{u}^{\mathrm{pw}} = \mathbf{u}_r^{\mathrm{pw}}\mathbf{e}_r + \mathbf{u}_\theta^{\mathrm{pw}}\,\mathbf{e}_\theta$ and $\mathbf{u} = \mathbf{u}_r\mathbf{e}_r + \mathbf{u}_\theta\,\mathbf{e}_\theta$. The boundary conditions are written as:

$$\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u}_r \;=\; -\,\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u}_r^{\mathrm{pw}}\,, \quad \mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u}_\theta \;=\; -\,\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u}_\theta^{\mathrm{pw}}\,, \quad \mathrm{p} \;=\; -\,\mathrm{p}^{\mathrm{pw}} \quad , \quad \partial\mathbb{B}_{(0,\mathfrak{a})}\,. \tag{8.7}$$

We expand the coefficient of each component in Fourier series. For the right hand-side,

$$\mathbf{u}_r^{\mathrm{pw}} = \sum_{k\in\mathbb{Z}} \mathbf{u}_{r,k}^{\mathrm{pw}}\,e^{\mathrm{i}\,k\,\theta}\,, \quad \mathbf{u}_\theta^{\mathrm{pw}} = \sum_{k\in\mathbb{Z}} \mathbf{u}_{\theta,k}^{\mathrm{pw}}\,e^{\mathrm{i}\,k\,\theta}\,, \quad \mathrm{p}^{\mathrm{pw}} = \sum_{k\in\mathbb{Z}} \mathrm{p}_k^{\mathrm{pw}}\,e^{\mathrm{i}\,k\,\theta}\,.$$

For the unknowns:

$$\mathbf{u}_r = \sum_{k\in\mathbb{Z}} \mathbf{u}_{r,k}\,e^{\mathrm{i}\,k\,\theta}\,, \quad \mathbf{u}_\theta = \sum_{k\in\mathbb{Z}} \mathbf{u}_{\theta,k}\,e^{\mathrm{i}\,k\,\theta}\,, \quad \mathrm{p}_k = \sum_{k\in\mathbb{Z}} \mathrm{p}_k\,e^{\mathrm{i}\,k\,\theta}\,.$$

Using (6.19) and (6.24), we have:

$$\begin{aligned}
\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u}_{r,k} &= a_k\,\mathsf{s}_\mathrm{P}^{-1}\,\omega\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_\mathrm{P}\,|\boldsymbol{x}|)e^{ik\theta} + b_k\,\mathsf{s}_\mathrm{B}^{-1}\,\omega\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_\mathrm{B}\,|\boldsymbol{x}|)e^{ik\theta} - c_k\,\mathsf{s}_\mathrm{S}^{-2}\,\frac{ik}{|\boldsymbol{x}|}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_\mathrm{S}\,|\boldsymbol{x}|)e^{ik\theta}\,,\\
\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u}_{\theta,k} &= a_k\,\mathsf{s}_\mathrm{P}^{-2}\,\frac{ik}{|\boldsymbol{x}|}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_\mathrm{P}\,|\boldsymbol{x}|)e^{ik\theta} + b_k\,\mathsf{s}_\mathrm{B}^{-2}\,\frac{ik}{|\boldsymbol{x}|}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_\mathrm{B}\,|\boldsymbol{x}|)e^{ik\theta} + c_k\,\mathsf{s}_\mathrm{S}^{-1}\,\omega\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_\mathrm{S}\,|\boldsymbol{x}|)e^{ik\theta}\,, \qquad (8.8)\\
\mathrm{p}_k &= -\,a_k\,M\left(\beta_\mathrm{P} + \alpha\right)\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_\mathrm{P}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} - b_k\,M\left(\beta_\mathrm{B} + \alpha\right)\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_\mathrm{P}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta}\,.
\end{aligned}$$

Imposing (8.7), we obtain the following linear system satisfied by a_k, b_k, c_k in each mode k:

$$\mathbb{A}_k^{\mathbf{u},\mathrm{p}}\begin{pmatrix} a_k\\ b_k\\ c_k\end{pmatrix} = \begin{pmatrix} -\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u}_{r,k}^{\mathrm{pw}}\\ -\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u}_{\theta,k}^{\mathrm{pw}}\\ -\mathrm{p}_k^{\mathrm{pw}}\end{pmatrix}\,, \tag{8.9}$$

where the coefficients matrix is defined as:

$$\mathbb{A}_k^{\mathbf{u},\mathrm{p}} = \begin{pmatrix} \mathsf{s}_\mathrm{P}^{-1}\,\omega\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_\mathrm{P}\,\mathfrak{a}) & \mathsf{s}_\mathrm{B}^{-1}\,\omega\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_\mathrm{B}\,\mathfrak{a}) & -\,\mathsf{s}_\mathrm{S}^{-2}\,\frac{ik}{\mathfrak{a}}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_\mathrm{S}\,\mathfrak{a})\\ \mathsf{s}_\mathrm{P}^{-2}\,\frac{ik}{\mathfrak{a}}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_\mathrm{P}\,\mathfrak{a}) & \mathsf{s}_\mathrm{B}^{-2}\,\frac{ik}{\mathfrak{a}}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_\mathrm{B}\,\mathfrak{a}) & \mathsf{s}_\mathrm{S}^{-1}\,\omega\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_\mathrm{S}\,\mathfrak{a})\\ -M\left(\beta_\mathrm{P} + \alpha\right)\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_\mathrm{P}\,\mathfrak{a}) & -\,M\left(\beta_\mathrm{B} + \alpha\right)\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_\mathrm{P}\,\mathfrak{a}) & 0 \end{pmatrix}\,. \tag{8.10}$$

8.3 Numerical tests

We show the imaginary part of the solid velocity $\mathbf{u}_x$ in figure 7.

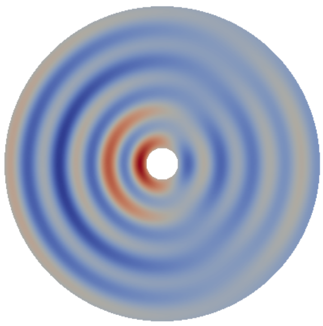
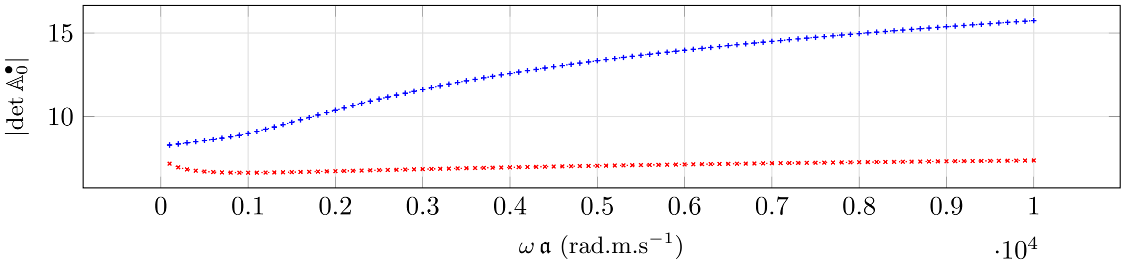


Figure 7: Scattering of a P plane wave on an impenetrable solid obstacle. Imaginary part of the solid velocity $\mathbf{u}_x$ of the reflected wave for boundary conditions of type 1 in a porous medium composed of inviscid sandstone with $f = 500$ Hz.

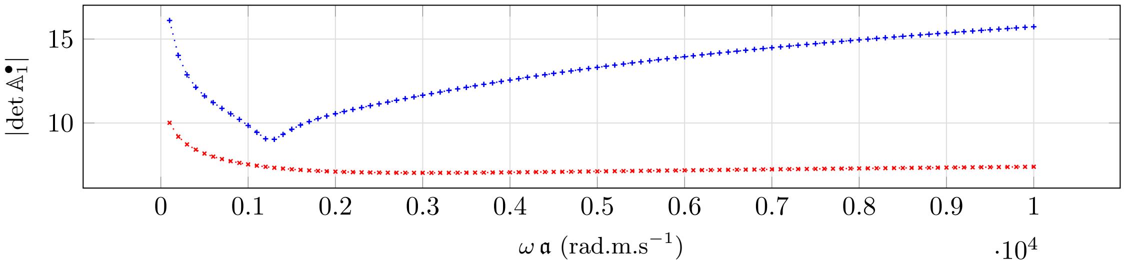
We will investigate the stability of the coefficient matrices $\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}$ (8.5) and $\mathbb{A}_k^{\mathbf{u},\mathrm{p}}$ (8.10) for the first modes k. The tests are divided in two parts. First, we consider a medium composed of sandstone with no viscosity, next we run the tests on a medium of sandstone with viscosity *cf.* 1. For both tests, the cross section radius is $\mathfrak{a} = 1$m. The results are reported in the following figures:

- Sandstone with no viscosity figures 8 and 9.
- Sandstone with viscosity figures 10 and 11.

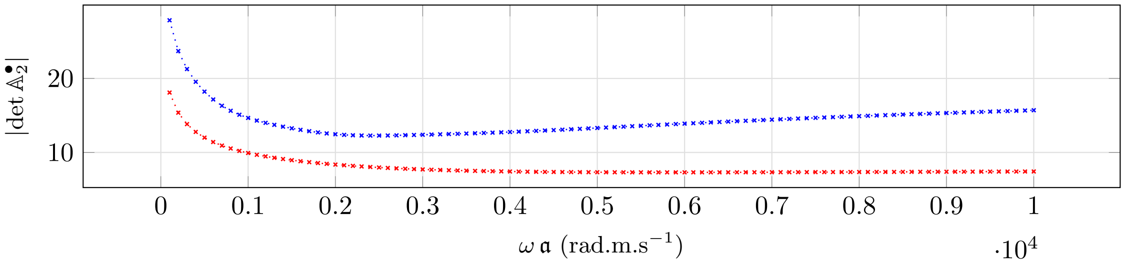
Due to the well-posedness of the problem, we expect no generalized eigenvalues in this case, which is clear for the curves of both boundary conditions (8,9, 10, 11). Note that the curves here are completely free of peaks, compared to the absence of generalized eigenvalues in the case in figure 3, in which the curves present small peaks. This corresponds to the true situation free of generalized eigenvalues.



(a) Mode $k = 0$

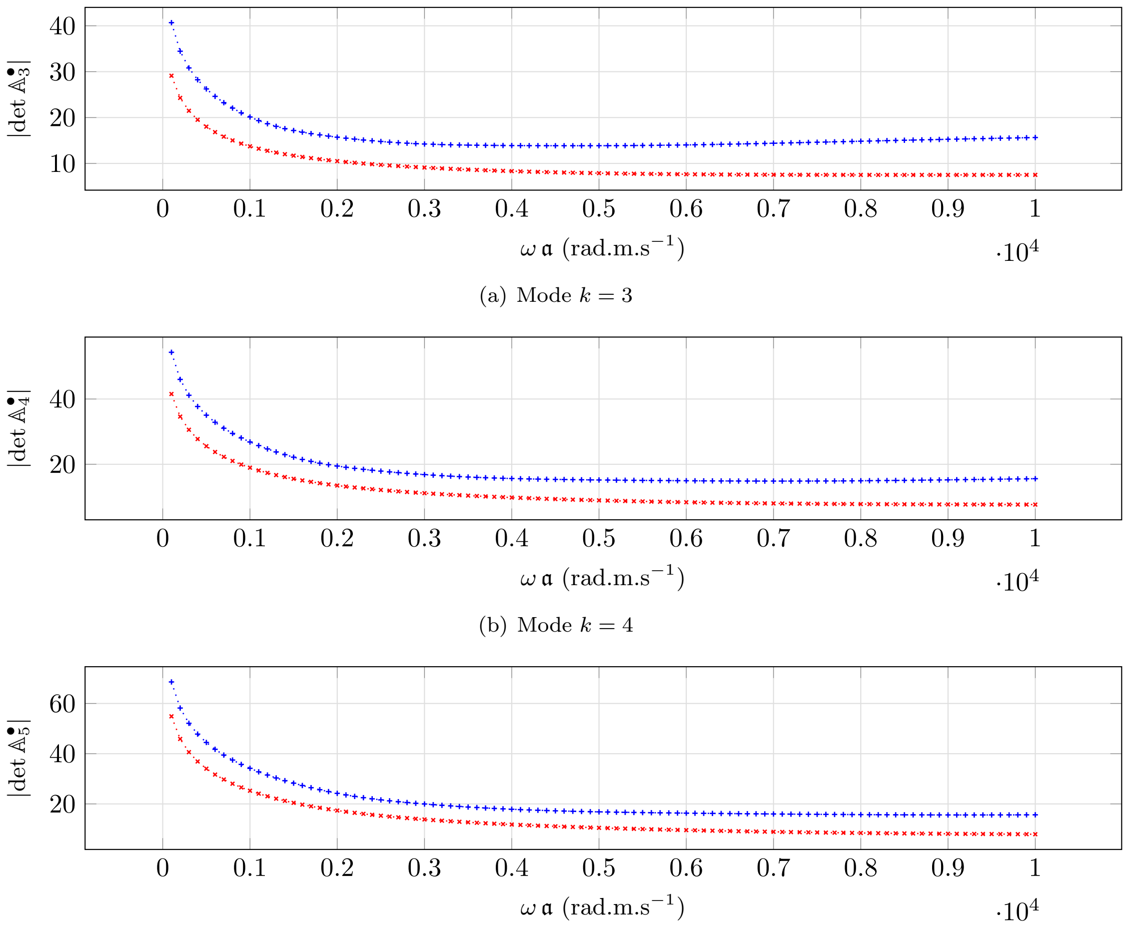


(b) Mode $k = 1$



(c) Mode $k = 2$

Figure 8: Experiment of a porous infinite medium with an impenetrable solid obstacle. Determinant of the coefficients matrix (log scale) for k in $0 : 2$ sandstone with no viscosity. The matrices corresponding with types of boundary conditions 1 and 3 are considered: $\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}$ (8.5) in blue and $\mathbb{A}_k^{\mathbf{u},\mathrm{p}}$ (8.10) in red .

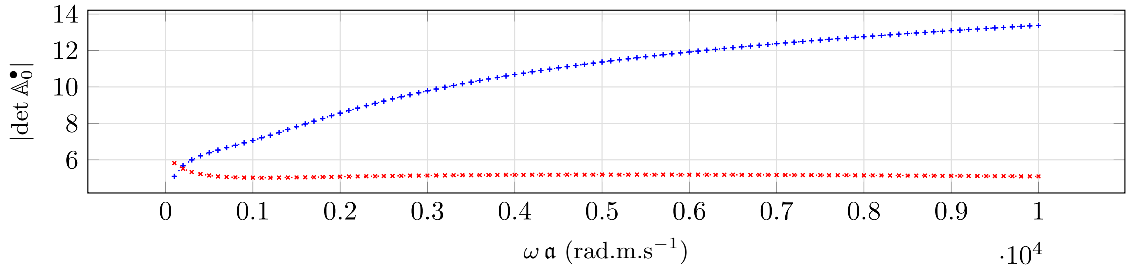


(a) Mode $k = 3$

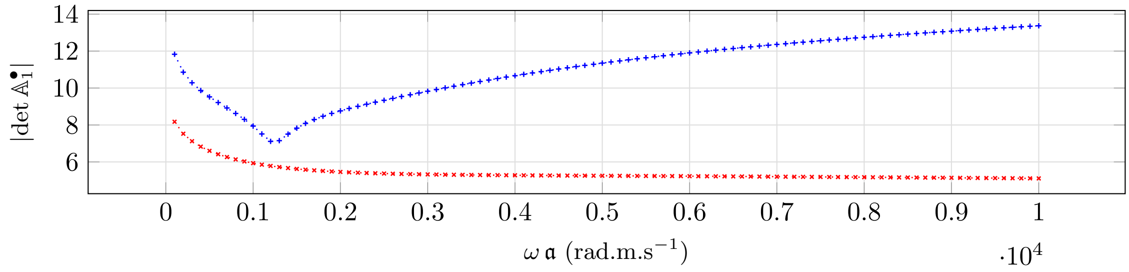
(b) Mode $k = 4$

(c) Mode $k = 5$

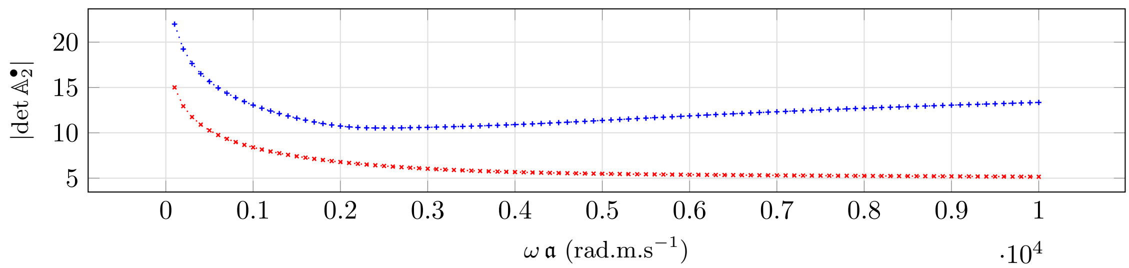
Figure 9: Experiment of a porous infinite medium with an impenetrable solid obstacle. Determinant of the coefficients matrix (log scale) for k in 3 : 5 for sandstone with no viscosity. The matrices corresponding with types of boundary conditions 1 and 3 are considered: $\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}$ (8.5) in blue and $\mathbb{A}_k^{\mathbf{u},\mathrm{p}}$ (8.10) in red .



(a) Mode $k = 0$

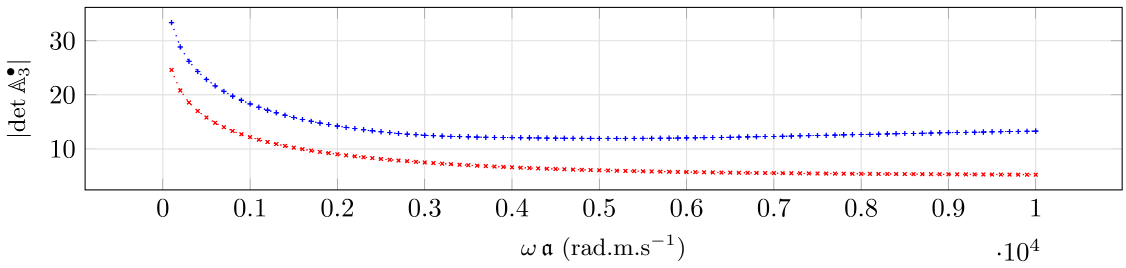


(b) Mode $k = 1$

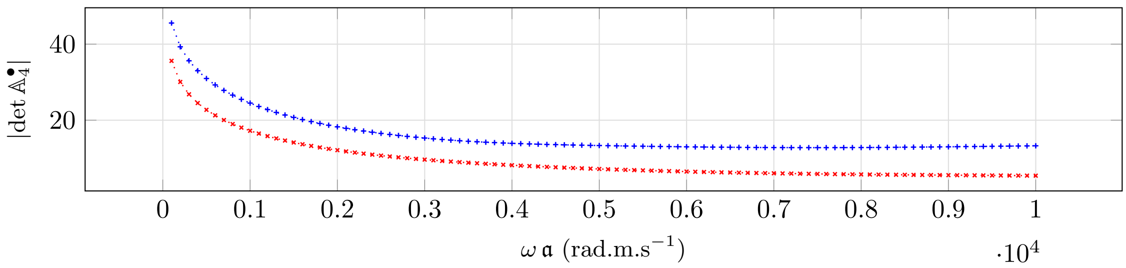


(c) Mode $k = 2$

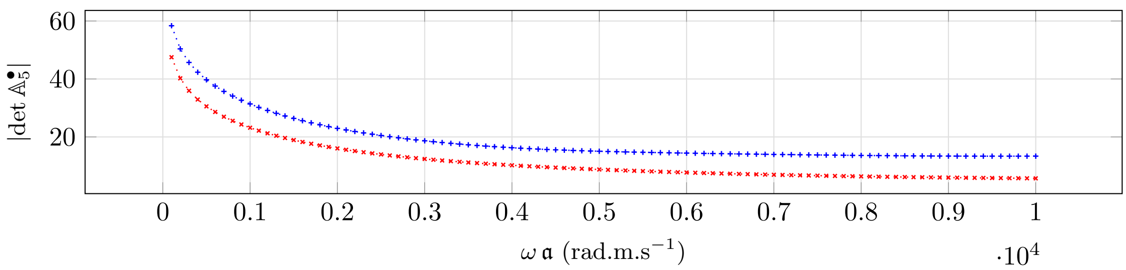
Figure 10: Experiment of a porous infinite medium with an impenetrable solid obstacle. Determinant of the coefficients matrix (log scale) for k in $0:2$ for a sandstone medium with viscosity $\eta \neq 0$. The matrices corresponding with types of boundary conditions 1 and 3 are considered: $\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}$ (8.5) in blue and $\mathbb{A}_k^{\mathbf{u},\mathrm{p}}$ (8.10) in red .



(a) Mode $k = 3$



(b) Mode $k = 4$



(c) Mode $k = 5$

Figure 11: Experiment of a porous infinite medium with an impenetrable solid obstacle. Determinant of the coefficients matrix (log scale) for k in $3 : 5$ for a sandstone medium with viscosity $\eta \neq 0$. The matrices corresponding with types of boundary conditions 1 and 3 are considered: $\mathbb{A}_k^{\mathbf{w},\boldsymbol{\tau}}$ (8.5) in blue and $\mathbb{A}_k^{\mathbf{u},\mathrm{p}}$ (8.10) in red.

9 Scattering of a plane wave by a penetrable porous solid inclusion immersed in a porous medium

Consider the scattering of a time-harmonic plane wave by a penetrable infinite cylinder immersed in another poroelastic infinite medium (see figure 12). The total wave outside of the cylinder is a superposition of the incident plane wave, and the reflected wave with each quantity satisfying poroelastic equations (3.30) in $\mathbb{R}^2 \setminus \mathbb{B}_{(0,\mathfrak{a})}$, while the transmitted wave is described by the displacement inside the cylinder. The unknown is now the reflected wave which is outgoing, and the transmitted wave. They are uniquely determined by transmission conditions imposed on the interface Γ. For $\bullet =$ pw, ref, trans, we denote:

$$U^\bullet = \begin{pmatrix} \mathbf{u}^\bullet \\ \mathbf{w}^\bullet \\ \boldsymbol{\tau}^\bullet \\ \mathrm{p}^\bullet \end{pmatrix}.$$

The unknowns U^{ref} and U^{trans} solve the following problem:

$$\begin{cases} U^{\text{ref}} \text{ solves the poroelastic equations (3.30) in } \mathbb{R}^2 \setminus \Omega; \\ U^{\text{trans}} \text{ solves the poroelastic equations (3.30) in } \Omega; \\ U^{\text{ref}} \text{ is outgoing;} \\ \text{Boundary conditions on the interface } \Gamma : \\ \quad \mathbf{u}^{\text{pw}} + \mathbf{u}^{\text{ref}} = \mathbf{u}^{\text{trans}}\,, \\ \quad \mathrm{p}^{\text{pw}} + \mathrm{p}^{\text{ref}} = \mathrm{p}^{\text{trans}}\,, \\ \quad \mathbf{w}^{\text{pw}} \cdot \boldsymbol{n} + \mathbf{w}^{\text{ref}} \cdot \boldsymbol{n} = \mathbf{w}^{\text{trans}} \cdot \boldsymbol{n}\,, \\ \quad \boldsymbol{\tau}^{\text{pw}} \cdot \boldsymbol{n} + \boldsymbol{\tau}^{\text{ref}} \cdot \boldsymbol{n} = \boldsymbol{\tau}^{\text{trans}} \cdot \boldsymbol{n}\,. \end{cases} \tag{9.1}$$

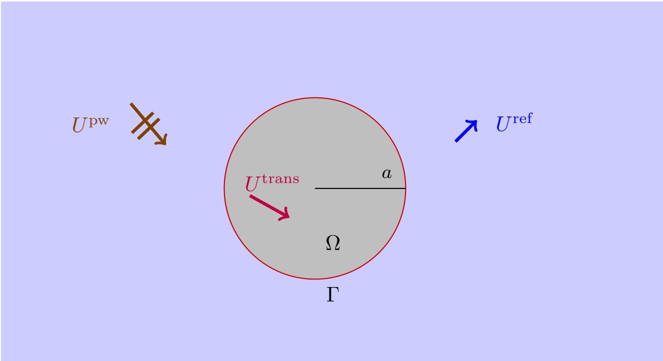


Figure 12: Scattering of a plane wave by a penetrable solid inclusion. The inclusion occupies the domain denoted by Ω. The cross section of the inclusion is a disc of radius denoted by a.

9.1 Construction of the analytical solution

The medium outside is denoted by medium 1, while the medium inside of the cylinder is denoted by medium 2. The slowness in the medium 1 is denoted by $\mathsf{s}_{\bullet,(I)}$ and in medium 2 by $\mathsf{s}_{\bullet,(II)}$.

The solutions $(\mathbf{u}, \mathbf{w}, \boldsymbol{\tau}, \mathrm{p})$ are given in the two media by equations (6.12), (6.13) and (6.14), while the potentials are given by (6.15). Hence, in medium 1, the potentials corresponding to the outgoing reflected wave satisfy equation (6.17):

$$\begin{aligned} \chi_{\mathrm{P,(I)}}(\boldsymbol{x}) &= \sum_{k\in\mathbb{Z}} a_k \, \mathrm{H}_k^{(1)}(\omega\, \mathsf{s}_{\mathrm{P,(I)}}|\boldsymbol{x}|)\; e^{\mathrm{i}\,k\,\theta}\,, \\ \chi_{\mathrm{B,(I)}}(\boldsymbol{x}) &= \sum_{k\in\mathbb{Z}} b_k \, \mathrm{H}_k^{(1)}(\omega\, \mathsf{s}_{\mathrm{B,(I)}}|\boldsymbol{x}|)\; e^{\mathrm{i}\,k\,\theta}\,, \\ \chi_{\mathrm{S,(I)}}(\boldsymbol{x}) &= \sum_{k\in\mathbb{Z}} c_k \, \mathrm{H}_k^{(1)}(\omega\, \mathsf{s}_{\mathrm{S,(I)}}|\boldsymbol{x}|)\; e^{\mathrm{i}\,k\,\theta}\,. \end{aligned}$$

Similarly , the potentials corresponding to the transmitted wave (*i.e.* in medium 2) are given by (6.16):

$$\begin{aligned} \chi_{\mathrm{P_{(II)}}}(\boldsymbol{x}) &= \sum_{k\in\mathbb{Z}} d_k \, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P,(II)}}|\boldsymbol{x}|)\; e^{\mathrm{i}\,k\,\theta}\,, \\ \chi_{\mathrm{B_{(II)}}}(\boldsymbol{x}) &= \sum_{k\in\mathbb{Z}} e_k \, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{B,(II)}}|\boldsymbol{x}|)\; e^{\mathrm{i}\,k\,\theta}\,, \\ \chi_{\mathrm{S_{(II)}}}(\boldsymbol{x}) &= \sum_{k\in\mathbb{Z}} f_k \, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{S,(II)}}|\boldsymbol{x}|)\; e^{\mathrm{i}\,k\,\theta}\,. \end{aligned}$$

Next, we are going to determine the coefficients a_k, b_k, c_k, d_k, e_k, f_k by imposing the boundary conditions at the interface between the two materials. Hence, they will satisfy the system given in equation (9.7). As previously, using the considered geometry, we express the unknowns in the polar basis:

$$\begin{aligned}
\mathbf{u}^{\bullet} &= \mathbf{u}^{\bullet}_r \mathbf{e}_r \;+\; \mathbf{u}^{\bullet}_\theta \mathbf{e}_\theta\,,\\
\mathbf{w}^{\bullet}\cdot\mathbf{n} &= \mathbf{w}^{\bullet}_r\,,\\
\boldsymbol{\tau}^{\bullet}\cdot\boldsymbol{n} &= \boldsymbol{\tau}^{\bullet}_{rr}\,\mathbf{e}_r + \boldsymbol{\tau}^{\bullet}_{r\theta}\,\mathbf{e}_\theta\,.
\end{aligned}$$

The boundary conditions are written as:

$$\begin{aligned}
\mathbf{u}_r^{\mathrm{ref}} - \mathbf{u}_r^{\mathrm{trans}} &= -\mathbf{u}_r^{\mathrm{pw}}\,,\\
\mathbf{u}_\theta^{\mathrm{ref}} - \mathbf{u}_\theta^{\mathrm{trans}} &= -\mathbf{u}_\theta^{\mathrm{pw}}\,,\\
\mathrm{p}^{\mathrm{ref}} - \mathrm{p}^{\mathrm{trans}} &= -\mathrm{p}^{\mathrm{pw}}\,,\\
\mathbf{w}_r^{\mathrm{ref}} - \mathbf{w}_r^{\mathrm{trans}} &= -\mathbf{w}_r^{\mathrm{pw}}\,,\\
\boldsymbol{\tau}_{rr}^{\mathrm{ref}} - \boldsymbol{\tau}_{rr}^{\mathrm{trans}} &= -\boldsymbol{\tau}_{rr}^{\mathrm{pw}}\,,\\
\boldsymbol{\tau}_{r\theta}^{\mathrm{ref}} - \boldsymbol{\tau}_{r\theta}^{\mathrm{trans}} &= -\boldsymbol{\tau}_{r\theta}^{\mathrm{pw}}\,.
\end{aligned} \tag{9.2}$$

The expansion of the coefficient of each component in Fourier series are:

$$\begin{aligned}
&\mathbf{u}^{\bullet}_r = \sum_{k\in\mathbb{Z}} \mathbf{u}^{\bullet}_{r,k}\, e^{\mathrm{i}\,k\,\theta}\,, \quad \mathbf{u}^{\bullet}_\theta = \sum_{k\in\mathbb{Z}} \mathbf{u}^{\bullet}_{\theta,k}\, e^{\mathrm{i}\,k\,\theta}\,, \quad \mathrm{p}^{\bullet} = \sum_{k\in\mathbb{Z}} \mathrm{p}^{\bullet}_k\, e^{\mathrm{i}\,k\,\theta}\,,\\
&\mathbf{w}^{\bullet}_r = \sum_{k\in\mathbb{Z}} \mathbf{w}^{\bullet}_{r,k}\, e^{\mathrm{i}\,k\,\theta}\,, \quad \boldsymbol{\tau}^{\bullet}_{rr} = \sum_{k\in\mathbb{Z}} \boldsymbol{\tau}^{\bullet}_{rr,k}\, e^{\mathrm{i}\,k\,\theta}\,, \quad \boldsymbol{\tau}^{\bullet}_{r\theta} = \sum_{k\in\mathbb{Z}} \boldsymbol{\tau}^{\bullet}_{r\theta,k}\, e^{\mathrm{i}\,k\,\theta}\,.
\end{aligned}$$

Using (6.19),(6.24), (6.20), (6.22) and (6.23), we have for the reflected wave:

$$\begin{aligned}
\mathfrak{s}\,\mathrm{i}\,\omega\mathbf{u}^{\mathrm{ref}}_{r,k} &= a_k\,\frac{\omega}{\mathsf{s}_{\mathrm{P},(\mathrm{I})}}\,\mathrm{H}^{(1)'}_k(\omega\,\mathsf{s}_{\mathrm{P},(\mathrm{I})}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta} + b_k\,\frac{\omega}{\mathsf{s}_{\mathrm{B},(\mathrm{I})}}\,\mathrm{H}^{(1)'}_k(\omega\,\mathsf{s}_{\mathrm{B},(\mathrm{I})}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta} - c_k\,\frac{\mathrm{i}k}{|\boldsymbol{x}|\mathsf{s}^2_{\mathrm{S},(\mathrm{I})}}\mathrm{H}^{(1)}_k(\omega\,\mathsf{s}_{\mathrm{S},(\mathrm{I})}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\,,\\
\mathfrak{s}\,\mathrm{i}\,\omega\mathbf{u}^{\mathrm{ref}}_{\theta,k} &= a_k\,\frac{\mathrm{i}k}{|\boldsymbol{x}|\,\mathsf{s}^2_{\mathrm{P},(\mathrm{I})}}\mathrm{H}^{(1)}_k(\omega\,\mathsf{s}_{\mathrm{P},(\mathrm{I})}|\boldsymbol{x}|)e^{\mathrm{i}k\theta} + b_k\,\frac{\mathrm{i}k}{|\boldsymbol{x}|\,\mathsf{s}^2_{\mathrm{B},(\mathrm{I})}}\mathrm{H}^{(1)}_k(\omega\,\mathsf{s}_{\mathrm{B},(\mathrm{I})}|\boldsymbol{x}|)e^{\mathrm{i}k\theta} + c_k\,\frac{\omega}{\mathsf{s}_{\mathrm{S},(\mathrm{I})}}\,\mathrm{H}^{(1)'}_k(\omega\,\mathsf{s}_{\mathrm{S},(\mathrm{I})}|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\,,\\
\mathrm{p}^{\mathrm{ref}}_k &= -\,a_k\,M\left(\beta_{\mathrm{P}} \;+\; \alpha\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P},(\mathrm{I})}\,|\boldsymbol{x}|)\,e^{\mathrm{i}k\theta} \;-\; b_k\,M\left(\beta_{\mathrm{B}} \;+\; \alpha\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B},(\mathrm{I})}\,|\boldsymbol{x}|)\,e^{\mathrm{i}k\theta}\,,\\
\mathfrak{s}\,\mathrm{i}\,\omega\mathbf{w}^{\mathrm{ref}}_{r,k} &= a_k\frac{\beta_{\mathrm{P}}\,\omega}{\mathsf{s}_{\mathrm{P},(\mathrm{I})}}\mathrm{H}^{(1)'}_k(\omega\,\mathsf{s}_{\mathrm{P},(\mathrm{I})}|\boldsymbol{x}|)e^{\mathrm{i}k\theta} + b_k\frac{\beta_{\mathrm{B}}\,\omega}{\mathsf{s}_{\mathrm{B},(\mathrm{I})}}\mathrm{H}^{(1)'}_k(\omega\,\mathsf{s}_{\mathrm{B},(\mathrm{I})}|\boldsymbol{x}|)e^{\mathrm{i}k\theta} + c_k\frac{\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}}\,\mathrm{i}k}{\det A|\boldsymbol{x}|}\,\mathrm{H}^{(1)}_k(\omega\,\mathsf{s}_{\mathrm{S},(\mathrm{I})}|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\,,
\end{aligned} \tag{9.3}$$

and

$$
\begin{aligned}
\omega^2 \boldsymbol{\tau}^{\mathrm{ref}}_{rr,k} = & -\frac{2\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{P,(I)}}\,|\boldsymbol{x}|} a_k\,\mathrm{H}^{(1)}_{k+1}(\omega\,\mathsf{s}_{\mathrm{P,(I)}}|\boldsymbol{x}|)e^{\mathrm{i}\,k\,\theta} + \frac{2\mu_{\mathrm{fr}}\,k}{\mathsf{s}^2_{\mathrm{P,(I)}}\,|\boldsymbol{x}|^2} a_k\,\mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{P,(I)}}|\boldsymbol{x}|)e^{\mathrm{i}\,k\,\theta} + 2\mu_{\mathrm{fr}}\,\omega^2 a_k\,\mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{P,(I)}}|\boldsymbol{x}|)e^{\mathrm{i}\,k\,\theta} \\
& -\frac{2\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}^2_{\mathrm{P,(I)}}\,|\boldsymbol{x}|^2} a_k\,\mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{P,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} - \frac{2\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{B,(I)}}\,|\boldsymbol{x}|} b_k\,\mathrm{H}^{(1)}_{k+1}(\omega\,\mathsf{s}_{\mathrm{B,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} + \frac{2\mu_{\mathrm{fr}}\,k}{\mathsf{s}^2_{\mathrm{B,(I)}}\,|\boldsymbol{x}|^2} b_k\,\mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{B,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} \\
& +2\mu_{\mathrm{fr}}\,\omega^2\,b_k \mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{B,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} - \frac{2\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}^2_{\mathrm{B,(I)}}\,|\boldsymbol{x}|^2}\,b_k\,\mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{B,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} + \frac{2\mu_{\mathrm{fr}}\,\omega\mathrm{i}k}{\mathsf{s}_{\mathrm{S,(I)}}|\boldsymbol{x}|}\;c_k\,\mathrm{H}^{(1)'}_{k}(\omega\,\mathsf{s}_{\mathrm{S,(I)}}|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} \\
& +\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\,M\beta_{\mathrm{P}}\right)\;a_k\,\mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{P,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} \\
& +\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\,M\beta_{\mathrm{B}}\right)\,b_k\,\mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{B,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta}\,, \\
\omega^2\,\boldsymbol{\tau}^{\mathrm{ref}}_{r\theta,k} = & -\frac{2\,\mu_{\mathrm{fr}}\omega\,\mathrm{i}\,k}{|\boldsymbol{x}|\,\mathsf{s}_{\mathrm{P,(I)}}}\,a_k\,\mathrm{H}^{(1)'}_{k}(\omega\,\mathsf{s}_{\mathrm{P,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} + \frac{2\,\mathrm{i}\,\mu_{\mathrm{fr}}\,k}{|\boldsymbol{x}|^2\,\mathsf{s}^2_{\mathrm{P,(I)}}}\,a_k\,\mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{P,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} \\
& -\frac{2\,\mu_{\mathrm{fr}}\omega\,\mathrm{i}\,k}{|\boldsymbol{x}|\,\mathsf{s}_{\mathrm{B,(I)}}}\,b_k\,\mathrm{H}^{(1)'}_{k}(\omega\,\mathsf{s}_{\mathrm{B,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} + \frac{2\,\mathrm{i}\,\mu_{\mathrm{fr}}\,k}{|\boldsymbol{x}|^2\,\mathsf{s}^2_{\mathrm{B,(I)}}}\,b_k\,\mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{B,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} - \frac{\mu_{\mathrm{fr}}\,k^2}{|\boldsymbol{x}|^2\,\mathsf{s}^2_{\mathrm{S}}}\,c_k\,\mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{S,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} \\
& +\frac{\mu_{\mathrm{fr}}\,\omega}{|\boldsymbol{x}|\,\mathsf{s}_{\mathrm{S,(I)}}}\,c_k\,\mathrm{H}^{(1)'}_{k}(\omega\,\mathsf{s}_{\mathrm{S,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} - \mu_{\mathrm{fr}}\,\frac{\omega}{\mathsf{s}_{\mathrm{S,(I)}}|\boldsymbol{x}|}\,c_k\,\mathrm{H}^{(1)}_{k+1}(\omega\,\mathsf{s}_{\mathrm{S,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} \\
& +\mu_{\mathrm{fr}}\,\frac{k}{\mathsf{s}^2_{\mathrm{S,(I)}}|\boldsymbol{x}|^2}c_k\,\mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{S,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} + \omega^2\,c_k\,\mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{S,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta} - \mu_{\mathrm{fr}}\,\frac{k^2}{\mathsf{s}^2_{\mathrm{S,(I)}}|\boldsymbol{x}|^2}\,c_k\,\mathrm{H}^{(1)}_{k}(\omega\,\mathsf{s}_{\mathrm{S,(I)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}\,k\,\theta}\,,
\end{aligned}
\tag{9.4}
$$

and for the transmitted wave:

$$
\begin{aligned}
\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u}^{\mathrm{trans}}_{r,k} &= a_k\,\frac{\omega}{\mathsf{s}_{\mathrm{P,(II)}}}\,\mathrm{J}'_k(\omega\,\mathsf{s}_{\mathrm{P,(II)}}|\boldsymbol{x}|)e^{\mathrm{i}k\theta} + b_k\,\frac{\omega}{\mathsf{s}_{\mathrm{B,(II)}}}\,\mathrm{J}'_k(\omega\,\mathsf{s}_{\mathrm{B,(II)}}\,|\boldsymbol{x}|)e^{\mathrm{i}k\theta} - c_k\,\frac{\mathrm{i}k}{|\boldsymbol{x}|\mathsf{s}^2_{\mathrm{S,(II)}}}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S,(II)}}|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\,, \\
\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{u}^{\mathrm{trans}}_{\theta,k} &= a_k\,\frac{\mathrm{i}k}{|\boldsymbol{x}|\,\mathsf{s}^2_{\mathrm{P,(II)}}}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P,(II)}}|\boldsymbol{x}|)e^{\mathrm{i}k\theta} + b_k\,\frac{\mathrm{i}k}{|\boldsymbol{x}|\,\mathsf{s}^2_{\mathrm{B,(II)}}}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B,(II)}}|\boldsymbol{x}|)e^{\mathrm{i}k\theta} + c_k\,\frac{\omega}{\mathsf{s}_{\mathrm{S,(II)}}}\,\mathrm{J}'_k(\omega\,\mathsf{s}_{\mathrm{S,(II)}}|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\,, \\
\mathrm{p}^{\mathrm{ref}}_k &= -\,a_k\,M\left(\beta_{\mathrm{P}} + \alpha\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P,(II)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}k\theta} - b_k\,M\left(\beta_{\mathrm{B}} + \alpha\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B,(II)}}\,|\boldsymbol{x}|)\,e^{\mathrm{i}k\theta}\,, \\
\mathfrak{s}\,\mathrm{i}\,\omega\,\mathbf{w}^{\mathrm{trans}}_{r,k} &= a_k\,\frac{\beta_{\mathrm{P}}\omega}{\mathsf{s}_{\mathrm{P,(II)}}}\mathrm{J}'_k(\omega\,\mathsf{s}_{\mathrm{P,(II)}}|\boldsymbol{x}|)e^{\mathrm{i}k\theta} + b_k\,\frac{\beta_{\mathrm{B}}\omega}{\mathsf{s}_{\mathrm{B,(II)}}}\mathrm{J}'_k(\omega\,\mathsf{s}_{\mathrm{B,(II)}}|\boldsymbol{x}|)e^{\mathrm{i}k\theta} + c_k\,\frac{\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}}\mathrm{i}k}{\mathrm{detA}|\boldsymbol{x}|}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S,(II)}}|\boldsymbol{x}|)e^{\mathrm{i}k\theta}\,,
\end{aligned}
\tag{9.5}
$$

and

$$
\begin{aligned}
\omega^2 \boldsymbol{\tau}^{\mathrm{trans}}_{rr,k} &= -\frac{2\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{P,(II)}}|\boldsymbol{x}|} a_k \mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{P,(II)}}|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} + \frac{2\mu_{\mathrm{fr}}\,k}{\mathsf{s}^2_{\mathrm{P,(II)}}|\boldsymbol{x}|^2} a_k \mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P,(II)}}|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} + 2\mu_{\mathrm{fr}}\,\omega^2\, a_k \mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P,(II)}}|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} \\
&- \frac{2\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}^2_{\mathrm{P,(I)}}|\boldsymbol{x}|^2} a_k\, \mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P,(II)}}|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} - \frac{2\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{B,(II)}}|\boldsymbol{x}|} b_k\, \mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{B,(II)}}|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} + \frac{2\mu_{\mathrm{fr}}\,k}{\mathsf{s}^2_{\mathrm{B,(II)}}|\boldsymbol{x}|^2} b_k\, \mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B,(II)}}|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} \\
&+ 2\mu_{\mathrm{fr}}\, b_k\,\omega^2\, \mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B,(II)}}|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} - \frac{2\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}^2_{\mathrm{B,(II)}}|\boldsymbol{x}|^2} b_k\, \mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B,(II)}}\,|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} + \frac{2\mu_{\mathrm{fr}}\omega\,\mathrm{i}k}{\mathsf{s}_{\mathrm{S,(II)}}|\boldsymbol{x}|} c_k\, \mathrm{J}'_k(\omega\,\mathsf{s}_{\mathrm{S,(II)}}\,|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} \\
&+ \omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\, M \beta_{\mathrm{P}}\right)\; a_k\, \mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P,(II)}}\,|\boldsymbol{x}|)\; e^{\mathrm{i}\,k\,\theta} \\
&+ \omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\, M \beta_{\mathrm{B}}\right)\, b_k\, \mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B,(II)}}\,|\boldsymbol{x}|)\; e^{\mathrm{i}\,k\,\theta}\,, \\
\omega^2 \boldsymbol{\tau}^{\mathrm{trans}}_{r\theta,k} &= -\frac{2\mu_{\mathrm{fr}}\omega\mathrm{i}k}{|\boldsymbol{x}|\mathsf{s}_{\mathrm{P,(II)}}} a_k \mathrm{J}'_k(\omega\mathsf{s}_{\mathrm{P,(II)}}|\boldsymbol{x}|) e^{\mathrm{i}k\theta} + \frac{2\,\mathrm{i}\,\mu_{\mathrm{fr}} k}{|\boldsymbol{x}|^2\mathsf{s}^2_{\mathrm{P,(II)}}} a_k \mathrm{J}_k(\omega\mathsf{s}_{\mathrm{P,(II)}}|\boldsymbol{x}|) e^{\mathrm{i}k\theta} - \frac{2\mu_{\mathrm{fr}}\omega\mathrm{i}k}{|\boldsymbol{x}|\mathsf{s}_{\mathrm{B,(II)}}} b_k \mathrm{J}'_k(\omega\,\mathsf{s}_{\mathrm{B,(II)}}|\boldsymbol{x}|) e^{\mathrm{i}k\theta} \\
&+ \frac{2\,\mathrm{i}\,\mu_{\mathrm{fr}}\,k}{|\boldsymbol{x}|^2\,\mathsf{s}^2_{\mathrm{B,(II)}}} b_k\, \mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B,(II)}}|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} - \frac{\mu_{\mathrm{fr}}\,k^2}{|\boldsymbol{x}|^2\,\mathsf{s}^2_{\mathrm{S}}} c_k\, \mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S,(II)}}|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} + \frac{\mu_{\mathrm{fr}}\,\omega}{|\boldsymbol{x}|\,\mathsf{s}_{\mathrm{S,(I)}}} c_k\, \mathrm{J}'_k(\omega\,\mathsf{s}_{\mathrm{S,(II)}}|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} \\
&- \mu_{\mathrm{fr}}\, \frac{\omega}{\mathsf{s}_{\mathrm{S,(II)}}|\boldsymbol{x}|}\, c_k\, \mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{S,(II)}}|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} + \mu_{\mathrm{fr}}\, \frac{k}{\mathsf{s}^2_{\mathrm{S,(iI)}}|\boldsymbol{x}|^2} c_k\, \mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S,(II)}}\,|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} \\
&+ \omega^2\, c_k\, \mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S,(II)}}|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta} - \mu_{\mathrm{fr}}\, \frac{k^2}{\mathsf{s}^2_{\mathrm{S,(I)}}|\boldsymbol{x}|^2}\, c_k\, \mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S,(II)}}|\boldsymbol{x}|)\, e^{\mathrm{i}\,k\,\theta}.
\end{aligned}
\tag{9.6}
$$

Imposing (9.2), we obtain a linear system satisfied by a_k, b_k, c_k, d_k, e_k, f_k in each mode k:

$$
\mathbb{A}^{\text{poro-poro}}_k \begin{pmatrix} a_k \\ b_k \\ c_k \\ d_k \\ e_k \\ f_k \end{pmatrix} = \begin{pmatrix} -\mathfrak{s}\,\mathrm{i}\,\omega\, \mathbf{u}^{\mathrm{pw}}_r \\ -\mathfrak{s}\,\mathrm{i}\,\omega\, \mathbf{u}^{\mathrm{pw}}_\theta \\ -\mathrm{p}^{\mathrm{pw}} \\ -\mathfrak{s}\,\mathrm{i}\,\omega\, \mathbf{w}^{\mathrm{pw}}_r \\ -\omega^2\, \boldsymbol{\tau}^{\mathrm{pw}}_{rr} \\ -\omega^2\, \boldsymbol{\tau}^{\mathrm{pw}}_{r\theta} \end{pmatrix}
\tag{9.7}
$$

with

$$
\mathbb{A}^{\text{poro-poro}}_k = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\
A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\
A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\
A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \\
A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} \\
A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66}
\end{pmatrix}
\tag{9.8}
$$

$$
\begin{aligned}
&A_{11} = \mathsf{s}_{\mathrm{P}}^{-1}\,\omega\, \mathrm{H}^{(1)'}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,, && A_{12} = \mathsf{s}_{\mathrm{B}}^{-1}\,\omega\, \mathrm{H}^{(1)'}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,, && A_{13} = -\,\mathsf{s}_{\mathrm{S}}^{-2}\,\frac{\mathrm{i}k}{\mathfrak{a}} \mathrm{H}^{(1)}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,, \\
&A_{21} = \mathsf{s}_{\mathrm{P}}^{-2}\,\frac{\mathrm{i}k}{\mathfrak{a}} \mathrm{H}^{(1)}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,, && A_{22} = \mathsf{s}_{\mathrm{B}}^{-2}\,\frac{\mathrm{i}k}{\mathfrak{a}} \mathrm{H}^{(1)}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,, && A_{23} = \mathsf{s}_{\mathrm{S}}^{-1}\,\omega\, \mathrm{H}^{(1)'}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,, \\
&A_{31} = -M\left(\beta_{\mathrm{P}} + \alpha\right) \mathrm{H}^{(1)}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,, && A_{32} = -\,M\left(\beta_{\mathrm{B}} + \alpha\right) \mathrm{H}^{(1)}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,, && A_{33} = 0\,,
\end{aligned}
\tag{9.9}
$$

$$
\begin{aligned}
&A_{14} = \mathsf{s}_{\mathrm{P}}^{-1}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,, && A_{15} = \mathsf{s}_{\mathrm{B}}^{-1}\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,, && A_{16} = -\,\mathsf{s}_{\mathrm{S}}^{-2}\,\frac{\mathrm{i}k}{\mathfrak{a}}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,,\\
&A_{24} = \mathsf{s}_{\mathrm{P}}^{-2}\,\frac{\mathrm{i}k}{\mathfrak{a}}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,, && A_{25} = \mathsf{s}_{\mathrm{B}}^{-2}\,\frac{\mathrm{i}k}{\mathfrak{a}}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,, && A_{26} = \mathsf{s}_{\mathrm{S}}^{-1}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,,\\
&A_{34} = -M\left(\beta_{\mathrm{P}} + \alpha\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,, && A_{35} = -\,M\left(\beta_{\mathrm{B}} + \alpha\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,, && A_{36} = 0\,,
\end{aligned}
\tag{9.10}
$$

$$
\begin{aligned}
&A_{41} = \frac{\beta_{\mathrm{P}}}{\mathsf{s}_{\mathrm{P}}}\,\omega\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a})\,, \qquad A_{42} = \frac{\beta_{\mathrm{B}}}{\mathsf{s}_{\mathrm{B}}}\,\omega\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{S}}\mathfrak{a})\,, \qquad A_{43} = \frac{\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}}}{\det A}\,\frac{\mathrm{i}k}{\mathfrak{a}}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\mathfrak{a})\,,\\
&A_{51} = -\,\frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{P}}\mathfrak{a}}\,\mathrm{H}_{k+1}^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a}) + \frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{P}}^2\mathfrak{a}^2}\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a}) + 2\,\mu_{\mathrm{fr}}\,\omega^2\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\\
&\qquad -\,\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{P}}^2\mathfrak{a}^2}\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a}) + \omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\,M\beta_{\mathrm{P}}\right)\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a})\,,\\
&A_{52} = -\,\frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{B}}\mathfrak{a}}\,\mathrm{H}_{k+1}^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a}) + \frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{B}}^2\mathfrak{a}^2}\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a}) + 2\,\mu_{\mathrm{fr}}\,\omega^2\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\\
&\qquad -\,\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{B}}^2\mathfrak{a}^2}\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a}) + \omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\,M\beta_{\mathrm{B}}\right)\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\mathfrak{a})\,,\\
&A_{53} = \frac{2\mu_{\mathrm{fr}}}{\mathsf{s}_{\mathrm{S}}\mathfrak{a}}\,\omega\,\mathrm{i}k\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{S}}\mathfrak{a})\,,
\end{aligned}
\tag{9.11}
$$

$$
\begin{aligned}
&A_{61} = -\frac{2\omega\,\mu_{\mathrm{fr}}\mathrm{i}k}{\mathfrak{a}\mathsf{s}_{\mathrm{P}}}\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a}) + \frac{2\,\mu_{\mathrm{fr}}\mathrm{i}k}{\mathfrak{a}^2\mathsf{s}_{\mathrm{P}}^2}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a})\,,\\
&A_{62} = -\frac{2\omega\,\mu_{\mathrm{fr}}\mathrm{i}k}{\mathfrak{a}\mathsf{s}_{\mathrm{B}}}\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{B}}\mathfrak{a}) + \frac{2\mu_{\mathrm{fr}}\mathrm{i}k}{\mathfrak{a}^2\mathsf{s}_{\mathrm{B}}^2}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\mathfrak{a})\,,\\
&A_{63} = -\,\frac{k^2\mu_{\mathrm{fr}}}{\mathfrak{a}^2\mathsf{s}_{\mathrm{S}}^2}\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a}) + \frac{\omega\,\mu_{\mathrm{fr}}}{\mathfrak{a}\mathsf{s}_{\mathrm{S}}}\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a}) - \frac{\omega}{\mathsf{s}_{\mathrm{S}}\mathfrak{a}}\,\mathrm{H}_{k+1}^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a}) + \frac{k}{\mathsf{s}_{\mathrm{S}}^2\mathfrak{a}^2}\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,,\\
&\qquad +\,\omega^2\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,e^{\mathrm{i}\,k\,\theta} - \frac{k^2}{\mathsf{s}_{\mathrm{S}}^2\mathfrak{a}^2}\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,,
\end{aligned}
\tag{9.12}
$$

$$
\begin{aligned}
&A_{44} = \frac{\beta_{\mathrm{P}}}{\mathsf{s}_{\mathrm{P}}}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a})\,, \qquad A_{45} = \frac{\beta_{\mathrm{B}}}{\mathsf{s}_{\mathrm{B}}}\,\omega\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\mathfrak{a})\,, \qquad A_{46} = \frac{\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}}}{\det A}\,\frac{\mathrm{i}k}{\mathfrak{a}}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\mathfrak{a})\,,\\
&A_{54} = -\,\frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{P}}\mathfrak{a}}\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a}) + \frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{P}}^2\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a}) + 2\,\mu_{\mathrm{fr}}\,\omega^2\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\\
&\qquad -\,\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{P}}^2\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a}) + \omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\,M\beta_{\mathrm{P}}\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\mathfrak{a})\,,\\
&A_{55} = -\,\frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{B}}\mathfrak{a}}\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,e^{\mathrm{i}\,k\,\theta} + \frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{B}}^2\mathfrak{a}^2}\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,e^{\mathrm{i}\,k\,\theta} + 2\,\mu_{\mathrm{fr}}\,\omega^2\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,e^{\mathrm{i}\,k\,\theta}\\
&\qquad -\,\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{B}}^2\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,e^{\mathrm{i}\,k\,\theta} + \omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\,M\beta_{\mathrm{B}}\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\mathfrak{a})\,,\\
&A_{56} = \frac{2\mu_{\mathrm{fr}}}{\mathsf{s}_{\mathrm{S}}\mathfrak{a}}\,\omega\,\mathrm{i}k\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\mathfrak{a})\,,
\end{aligned}
\tag{9.13}
$$

$$
\begin{aligned}
A_{64} &= -\frac{2\omega\,\mu_{\rm fr}\mathrm{i}k}{\mathfrak{a}\mathsf{s}_{\rm P}}\mathrm{J}_k'(\omega\,\mathsf{s}_{\rm P}\mathfrak{a}) + \frac{2\,\mu_{\rm fr}\mathrm{i}k}{\mathfrak{a}^2\mathsf{s}_{\rm P}^2}\mathrm{J}_k(\omega\,\mathsf{s}_{\rm P}\mathfrak{a})\,,\\
A_{65} &= -\frac{2\omega\,\mu_{\rm fr}\mathrm{i}k}{\mathfrak{a}\mathsf{s}_{\rm B}}\mathrm{J}_k'(\omega\,\mathsf{s}_{\rm B}\mathfrak{a}) + \frac{2\mu_{\rm fr}\mathrm{i}k}{\mathfrak{a}^2\mathsf{s}_{\rm B}^2}\mathrm{J}_k(\omega\,\mathsf{s}_{\rm B}\mathfrak{a})\,,\\
A_{66} &= -\frac{k^2\mu_{\rm fr}}{\mathfrak{a}^2\mathsf{s}_{\rm S}^2}\mathrm{J}_k(\omega\,\mathsf{s}_{\rm S}\,\mathfrak{a}) + \frac{\omega\,\mu_{\rm fr}}{\mathfrak{a}\mathsf{s}_{\rm S}}\mathrm{J}_k'(\omega\,\mathsf{s}_{\rm S}\,\mathfrak{a}) \;-\; \frac{\omega}{\mathsf{s}_{\rm S}\mathfrak{a}}\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\rm S}\,\mathfrak{a})\;e^{\mathrm{i}\,k\,\theta} \;+\; \frac{k}{\mathsf{s}_{\rm S}^2\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\rm S}\,\mathfrak{a})\;,\\
&\quad + \omega^2\,\mathrm{J}_k(\omega\,\mathsf{s}_{\rm S}\,\mathfrak{a})\;e^{\mathrm{i}\,k\,\theta} \;-\; \frac{k^2}{\mathsf{s}_{\rm S}^2\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\rm S}\,\mathfrak{a})\;.
\end{aligned}
\tag{9.14}
$$

We define the eigenvalues as follows:

Definition 5. ω is porous-porous transmission eigenvalue if $\det \mathbb{A}_k^{\text{poro-poro}}(\omega) = 0$, where $\mathbb{A}_k^{\text{poro-poro}}$ is the coefficients matrix defined in equation (9.8).

9.2 Numerical tests

We consider an infinite porous medium denoted as the exterior medium Γ, in which Ω is a porous inclusion (interior medium), see figure 12. We show the imaginary part of the solid velocity $\mathbf{u}_x$ in figure 13.

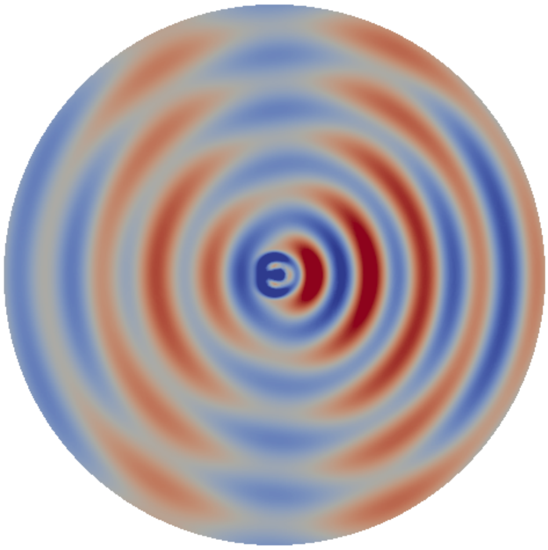


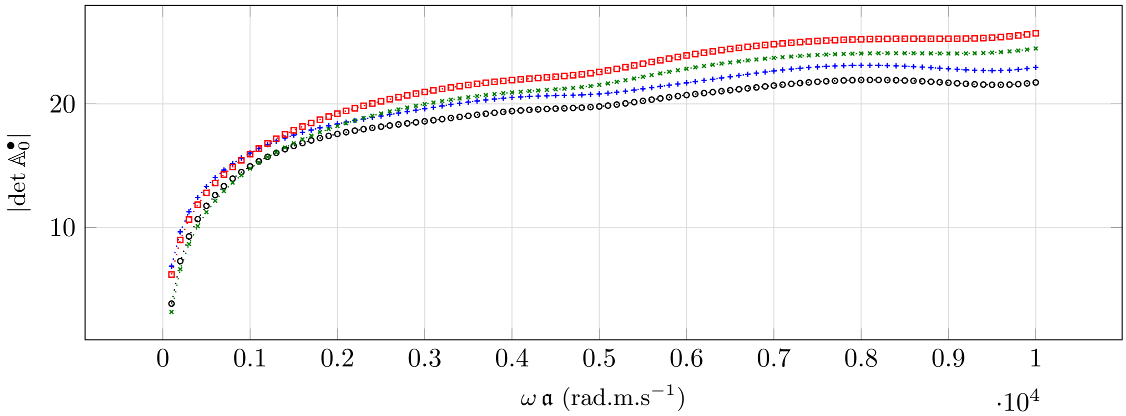
Figure 13: Scattering of a P plane wave by a penetrable solid inclusion. Total solution of the imaginary part of the solid velocity $\mathbf{u}_x$ for sandstone/shale with no viscosity test with $\omega = 500$ rad.s^{-1}.

To investigate the influence of the material parameters, we consider different cases detailled in the table 3. For all of these tests, we study the determinant of the coefficients matrix (9.8) function of the pulsation. We first study the influence of the viscosity by varying its value in the interior and exterior medium. Next, we vary the value of the frame shear modulus to observe the differences on the shape of the curves. For all tests, the cross section radius is $\mathfrak{a} = 1$m. We have the following observations:

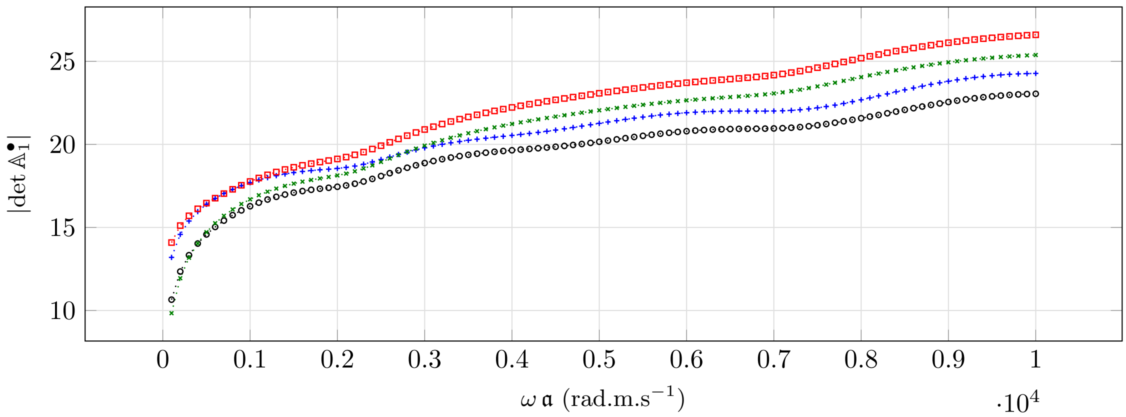
- As in the case of a bounded domain, the shear frame modulus has an influence on the shape of the curve. When the shear frame modulus of the interior material decreases, we can observe the apparition of smooth peaks. When it is in the exterior material, the general value of the determinant decreases, *cf.* 16. This shows that the interior medium has more influence on the determinant of the matrix than the exterior one.
- From mode 3, we can observe for every case that the red curve in figure 15, the one with viscosity in the exterior medium and no viscosity in the interior medium is higher than the other one for low frequencies.

Exterior medium	Interior medium	Figures
Sandstone ($\eta = 0$ Pa.s and $\mu_{\text{fr}} = 12$GPa)	Shale ($\eta = 0$ Pa.s and $\mu_{\text{fr}} = 3.96$GPa)	14, 15, 16
Sandstone ($\eta = 0$ Pa.s and $\mu_{\text{fr}} = 12$GPa)	Shale ($\eta = 10^{-3}$ Pa.s and $\mu_{\text{fr}} = 3.96$GPa)	14, 15
Sandstone ($\eta = 10^{-3}$Pa.s and $\mu_{\text{fr}} = 12$GPa)	Shale ($\eta = 0$ Pa.s and $\mu_{\text{fr}} = 3.96$GPa)	14, 15
Sandstone ($\eta = 10^{-3}$Pa.s and $\mu_{\text{fr}} = 12$GPa)	Shale ($\eta = 10^{-3}$ Pa.s and $\mu_{\text{fr}} = 3.96$GPa)	14, 15
Sandstone ($\eta = 0$ Pa.s and $\mu_{\text{fr}} = 12$GPa)	Modified Shale ($\eta = 0$ Pa.s and $\mu_{\text{fr}} = 1$GPa)	16,
Modified Sandstone ($\eta = 0$ Pa.s and $\mu_{\text{fr}} = 3$GPa)	Shale ($\eta = 0$ Pa.s and $\mu_{\text{fr}} = 3.96$GPa)	16
Modified Sandstone ($\eta = 0$ Pa.s and $\mu_{\text{fr}} = 3$GPa)	Modified Shale ($\eta = 0$ Pa.s and $\mu_{\text{fr}} = 1$GPa)	16

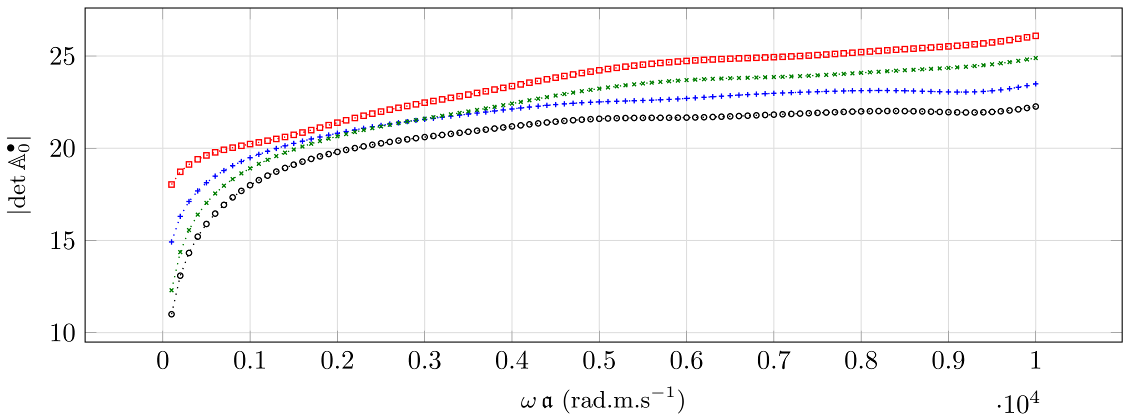
Table 3: List of the tests of porous-porous interaction



(a) Mode $k = 0$

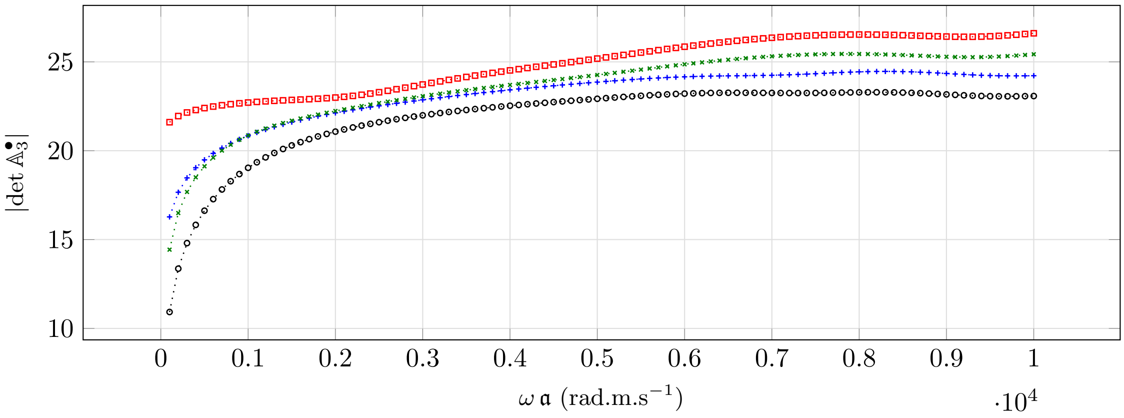


(b) Mode $k = 1$

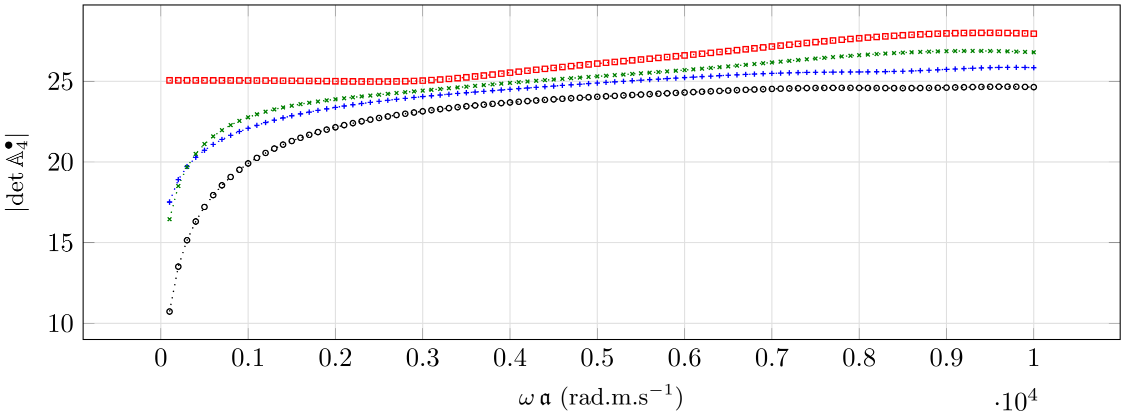


(c) Mode $k = 2$

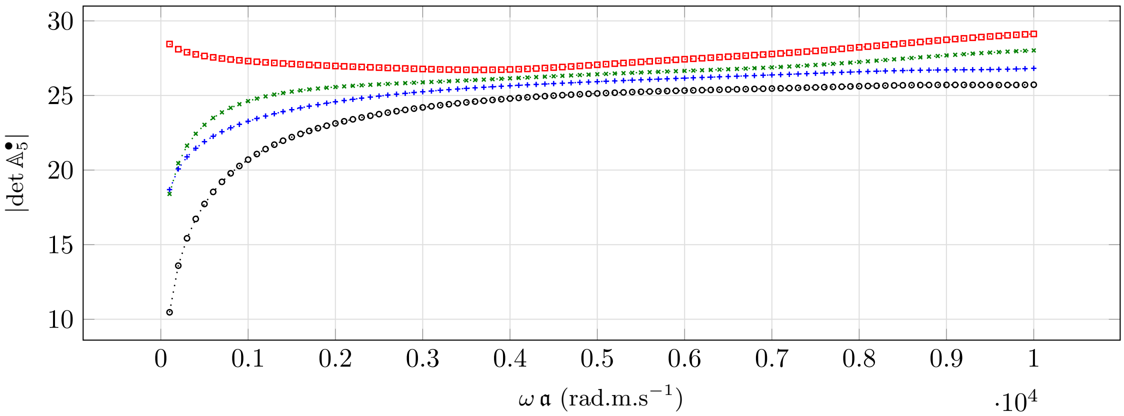
Figure 14: Experiment for a porous-porous interaction: Determinant of the coefficients matrix $\mathbb{A}_k^{\text{poro-poro}}$ (9.8) (log scale) for k in $0:2$ for sandstone/shale with no viscosity in blue ⋯□⋯, for inviscid sandstone/ viscous shale in red ⋯○⋯ , for viscous sandstone/inviscid shale in black ⋯×⋯ and for viscous sandstone/viscous shale in green ⋯+⋯.



(a) Mode $k = 3$

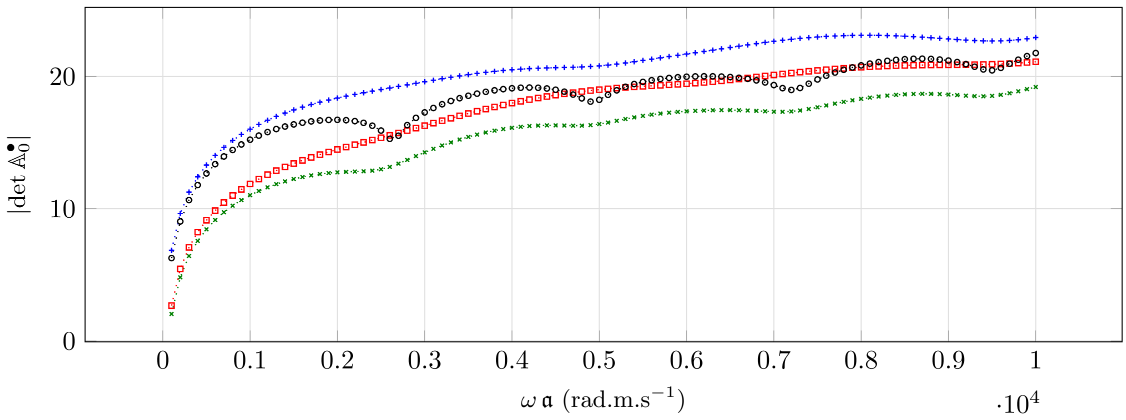


(b) Mode $k = 4$



(c) Mode $k = 5$

Figure 15: Experiment for a porous-porous interaction: Determinant of the coefficients matrix $\mathbb{A}_k^{\text{poro-poro}}$ (9.8) (log scale) for k in $3:5$ for sandstone/shale with no viscosity in blue, for inviscid sandstone/ viscous shale in red, for viscous sandstone/inviscid shale in black and for viscous sandstone/viscous shale in green.



(a) Modes $k = 0$

Figure 16: Experiment for a porous-porous interaction: Comparison of the determinant of the coefficients matrix $\mathbb{A}_k^{\text{poro-poro}}$ (9.8) (log scale) for $k = 0$ for sandstone/shale with no viscosity and regular shear frame modulus in blue ⋯∘⋯. The shear frame modulus of the interior material is divided by four in ⋯▫⋯, the one for the exterior material is divided by four in ⋯×⋯ and both are divided by four in +.

10 Scattering of a plane wave by a poroelastic domain in a fluid medium

We consider the scattering by a plane wave of a poroelastic obstacle in an infinite fluid medium. The total wave outside of the cylinder is a superposition of the incident plane wave and the reflected wave, while the transmitted wave is described by the displacement inside the cylinder. The movement in the fluid region is described by

$$\begin{aligned} \mathrm{p}_{\text{flu}} &= \mathrm{p}^{\text{pw}} + \mathrm{p}^{\text{ref}}\,, \\ \mathbf{u}_{\text{flu}} &= \mathbf{u}^{\text{pw}} + \mathbf{u}^{\text{ref}}\,. \end{aligned} \tag{10.1}$$

where $(\mathrm{p}_{\bullet}, \mathbf{u}_{\bullet})$ satisfy the Helmholtz equation for p,

$$\left(-\Delta \,-\, \omega^2\, \mathsf{s}_{\text{flu}}^2\right) \mathrm{p}_{\text{flu}} \,=0\,, \tag{10.2}$$

and the velocity in fluid is given by

$$\mathbf{u}_{\text{flu}} = -\,\frac{1}{\rho_{\text{flu}}\,\mathfrak{s}\,\mathrm{i}\,\omega}\nabla\,\mathrm{p}_{\text{flu}} \;=\; \frac{\mathfrak{s}\,\mathrm{i}}{\rho_{\text{flu}}\,\omega}\nabla\,\mathrm{p}_{\text{flu}}\,. \tag{10.3}$$

Here, slowness of fluid is chosen in the same way as those in the poroelastic interior by Definition (5.33) , i.e.

$$\mathsf{s}_{\text{flu}} \,=\, -\mathfrak{s}\,\sqrt{\mathsf{s}_{\text{flu}}^2}\,. \tag{10.4}$$

In the interior, the transmitted movements are described by $U^{\text{trans}} \,=\, (\mathbf{u}^{\text{trans}}\,, \mathbf{w}^{\text{trans}}\,, \boldsymbol{\tau}^{\text{trans}}\,, \mathrm{p}^{\text{trans}})$ that solves the poroelastic equations (3.30). The interior and exterior quantities are determined by transmission conditions imposed on the interface Γ.

In short, the unknowns of the fluid-solid interaction problem are $(\mathrm{p}_{\mathrm{flu}}, \mathbf{u}_{\mathrm{flu}})$ and U^{trans}, which solve:

$$\begin{cases} (\mathrm{p}_{\mathrm{flu}}, \mathbf{u}_{\mathrm{flu}}) \text{ solves the acoustic equations (10.2) and (10.3) in } \mathbb{R}^2 \setminus \Omega; \\ U^{\mathrm{trans}} \text{ solves the poroelastic equations (3.30) in } \Omega; \\ (\mathrm{p}_{\mathrm{flu}}, \mathbf{u}_{\mathrm{flu}}) \text{ are outgoing in the sense of } \lim_{r\to\infty} \sqrt{r}\left(\dfrac{\partial \mathrm{p}_{\mathrm{flu}}}{\partial r} - \mathrm{i}\,\omega\,\mathsf{s}_{\mathrm{flu}}\,\mathrm{p}_{\mathrm{flu}}\right) = 0\,; \\ \text{Boundary conditions on the interface } \Gamma : (3.37) \\ \quad (\mathbf{u}_{\mathrm{flu}} - \mathbf{u}) \cdot \boldsymbol{n} = \mathbf{w} \cdot \boldsymbol{n}\,, \\ \quad \mathrm{p}_{\mathrm{flu}} - \mathrm{p} = \dfrac{1}{\kappa_\Gamma}\,\mathbf{w} \cdot \boldsymbol{n}\,, \\ \quad \boldsymbol{\tau} \cdot \boldsymbol{n} = -\,\mathrm{p}_{\mathrm{flu}} \cdot \boldsymbol{n}\,. \end{cases} \tag{10.5}$$

where κ_Γ denotes the hydraulic permeability on the interface.

We will distinguish three different cases for κ_Γ, we first consider a finite value of κ_Γ in 10.1.1. Then when $\kappa_\Gamma \to \infty$, the pores are open, and the second condition in (10.5) becomes $\mathrm{p}_{\mathrm{flu}} - \mathrm{p} = 0$. This case is detailed in subsection 10.1.2. We finally study the case of sealed pores in section 10.1.3. This means that $\kappa_\Gamma = 0$, and the second interface condition is modified as $\mathbf{w} \cdot \boldsymbol{n} = 0$.

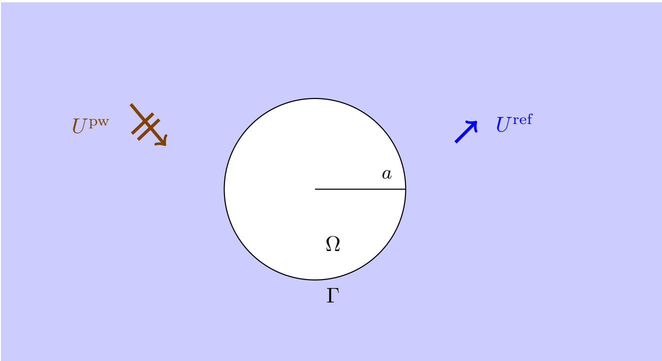


Figure 17: Scattering of a fluid plane wave by a poroelastic inclusion. The inclusion occupies the domain denoted by Ω. The cross section of the inclusion is a disc of radius denoted by $\mathfrak{a}$.

10.1 Construction of the analytical solution

In the fluid, the pressure and the velocities are expressed as follows:

$$\begin{aligned} \mathrm{p}_{\mathrm{flu}} &= \mathrm{p}^{\mathrm{pw}} + \mathrm{p}^{\mathrm{ref}}\,, \\ \mathbf{u}_{\mathrm{flu}} &= \mathbf{u}^{\mathrm{pw}} + \mathbf{u}^{\mathrm{ref}}\,. \end{aligned} \tag{10.6}$$

The pressure $\mathrm{p}_{\mathrm{flu}}$ satisfies the Helmholtz equation. In the fluid,

$$\mathbf{u}_{\mathrm{flu}} = -\frac{1}{\rho_{\mathrm{flu}}\,\mathfrak{s}\,\mathrm{i}\,\omega}\nabla\,\mathrm{p}_{\mathrm{flu}} = \frac{\mathfrak{s}\,\mathrm{i}}{\rho_{\mathrm{flu}}\,\omega}\nabla\,\mathrm{p}_{\mathrm{flu}}\,.$$

The incident plane wave is

$$\begin{aligned} \mathrm{p}^{\mathrm{pw}}_{\mathrm{flu}} &= \sum_{k=-\infty}^{\infty} \mathrm{i}^k\,J_k(\mathfrak{s}\,\omega\,\mathsf{s}_{\mathrm{flu}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\,, \\ \Rightarrow\ \mathbf{u}^{\mathrm{pw}}_{\mathrm{flu}} &= \sum_{k=-\infty}^{\infty} \frac{\mathfrak{s}\,\mathrm{i}^{k+1}}{\rho_{\mathrm{flu}}\,\omega}\,\nabla\,J_k(\mathfrak{s}\,\omega\,\mathsf{s}_{\mathrm{flu}}\,|\boldsymbol{x}|)\;e^{\mathrm{i}\,k(\theta-\alpha_{\mathrm{inc}})}\,. \end{aligned}$$

The reflected wave is written as:

$$\mathrm{p}_{\text{flu}}^{\text{ref}} = \sum_{k\in\mathbb{Z}} d_k\, \mathrm{H}_k^{(1)}(\omega\, \mathsf{s}_{\text{flu}}\, |\boldsymbol{x}|)\; e^{\mathrm{i}\, k\,\theta}\,,$$

$$\Rightarrow\; \mathbf{u}_{\text{flu}}^{\text{ref}} = \sum_{k\in\mathbb{Z}} \frac{\mathfrak{s}\,\mathrm{i}}{\rho_{\text{flu}}\,\omega} d_k\, \nabla\, \mathrm{H}_k^{(1)}(\omega\, \mathsf{s}_{\text{flu}}\, |\boldsymbol{x}|)\; e^{\mathrm{i}\, k\,\theta}\,.$$

In polar coordinates, using $\nabla = \partial_r \mathbf{e}_r + \dfrac{1}{|\boldsymbol{x}|}\partial_\theta \mathbf{e}_\theta$, the radial component is:

$$\mathbf{u}_{\text{flu},r}^{\text{pw}} = \sum_{k\in\mathbb{Z}} \frac{\mathsf{s}_{\text{flu}}}{\rho_{\text{flu}}}\, \mathrm{i}^{k+1}\, J_k'(\mathfrak{s}\,\omega\, \mathsf{s}_{\text{flu}}\, |\boldsymbol{x}|)\; e^{\mathrm{i}\, k(\theta-\alpha_{\text{inc}})}\,,$$

$$\mathbf{u}_{\text{flu},r}^{\text{ref}} = \sum_{k\in\mathbb{Z}} \frac{\mathfrak{s}\,\mathsf{s}_{\text{flu}}}{\rho_{\text{flu}}}\, \mathrm{i}\, d_k\, \mathrm{H}_k^{(1)'}(\omega\, \mathsf{s}_{\text{flu}}\, |\boldsymbol{x}|)\; e^{\mathrm{i}\, k\,\theta}\,.$$

In the poroelastic domain Ω, the potentials and the expressions of the unknowns are given in section 7 equations (7.4) and (7.10).

10.1.1 Finite positive value of hydraulic permeability

On a disc, $\boldsymbol{n} = \mathbf{e}_r$. Imposing the transmission condition (3.37) means that for every mode k:

$$\begin{aligned}
&\frac{\mathsf{s}_{\text{flu}}}{\rho_{\text{flu}}}\, \mathrm{i}^{k+1}\, J_k'(\omega\, \mathsf{s}_{\text{flu}}\, |\boldsymbol{x}|)\; e^{-\mathrm{i}\, k\alpha_{\text{inc}}} \;+\; \frac{\mathfrak{s}\,\mathsf{s}_{\text{flu}}}{\rho_{\text{flu}}}\, \mathrm{i}\, d_k\, \mathrm{H}_k^{(1)'}(\omega\, \mathsf{s}_{\text{flu}}\, |\boldsymbol{x}|)\\
&+\; a_k\, \mathfrak{s}\, \frac{\mathrm{i}}{\mathsf{s}_{\mathrm{P}}}\, \mathrm{J}_k'(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) \;+\; b_k\, \mathfrak{s}\, \frac{\mathrm{i}}{\mathsf{s}_{\mathrm{B}}}\, \mathrm{J}_k'(\omega\, \mathsf{s}_{\mathrm{B}}\, \mathfrak{a}) \;+\; c_k \frac{\mathfrak{s}\, k}{\mathfrak{a}\, \mathsf{s}_{\mathrm{S}}^2\, \omega} \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{S}}\, \mathfrak{a})\\
&= -\, a_k\, \mathfrak{s}\, \frac{\mathrm{i}\, \beta_{\mathrm{P}}}{\mathsf{s}_{\mathrm{P}}}\, \mathrm{J}_k'(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) \;-\; b_k\, \mathfrak{s}\, \frac{\mathrm{i}\, \beta_{\mathrm{B}}}{\mathsf{s}_{\mathrm{B}}}\, \mathrm{J}_k'(\omega\, \mathsf{s}_{\mathrm{B}}\, \mathfrak{a}) \;+\; c_k\, \frac{\rho_{\mathrm{f}}\, \mu_{\text{fr}}}{\det A}\, \frac{\mathfrak{s}\, k}{\mathfrak{a}\, \omega} \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{S}}\, \mathfrak{a})\,,
\end{aligned}$$

$$\begin{aligned}
&d_k\, \mathrm{H}_k^{(1)}(\omega\, \mathsf{s}_{\text{flu}}\, \mathfrak{a}) \;+\; \mathrm{i}^k\, J_k(\omega\, \mathsf{s}_{\text{flu}}\, \mathfrak{a})\; e^{-\mathrm{i}\, k\alpha_{\text{inc}}}\\
&+\; a_k\, M\left(\beta_{\mathrm{P}} \;+\; \alpha\right) \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) \;+\; b_k\, M\left(\beta_{\mathrm{B}} \;+\; \alpha\right) \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a})\\
&= -\, a_k\, \mathfrak{s}\, \frac{\mathrm{i}\, \beta_{\mathrm{P}}}{\mathsf{s}_{\mathrm{P}}\, \kappa_\Gamma}\, \mathrm{J}_k'(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) \;-\; b_k\, \mathfrak{s}\, \frac{\mathrm{i}\, \beta_{\mathrm{B}}}{\mathsf{s}_{\mathrm{B}}\, \kappa_\Gamma}\, \mathrm{J}_k'(\omega\, \mathsf{s}_{\mathrm{B}}\, \mathfrak{a}) \;+\; c_k\, \frac{\rho_{\mathrm{f}}\, \mu_{\text{fr}}}{\det A}\, \frac{\mathfrak{s}\, k}{\mathfrak{a}\, \omega\, \kappa_\Gamma} \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{S}}\, \mathfrak{a})\,,
\end{aligned}$$

$$\begin{aligned}
&-\frac{2\,\mu_{\text{fr}}\,\omega}{\mathsf{s}_{\mathrm{P}}\mathfrak{a}}\, a_k\, \mathrm{J}_{k+1}(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) \;+\; \frac{2\,\mu_{\text{fr}}\, k}{\mathsf{s}_{\mathrm{P}}^2\mathfrak{a}^2}\, a_k\, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) \;+2\,\mu_{\text{fr}}\, a_k\, \omega^2\, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a})\\
&-\frac{2\,\mu_{\text{fr}}\, k^2}{\mathsf{s}_{\mathrm{P}}^2\mathfrak{a}^2}\, a_k\, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) \;-\; \frac{2\,\mu_{\text{fr}}\,\omega}{\mathsf{s}_{\mathrm{B}}\mathfrak{a}}\, b_k\, \mathrm{J}_{k+1}(\omega\, \mathsf{s}_{\mathrm{B}}\, \mathfrak{a}) \;+\; \frac{2\,\mu_{\text{fr}}\, k}{\mathsf{s}_{\mathrm{B}}^2\mathfrak{a}^2}\, b_k\, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{B}}\, \mathfrak{a})\\
&+2\,\mu_{\text{fr}}\, b_k\, \omega^2\, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{B}}\, \mathfrak{a}) \;-\; \frac{2\,\mu_{\text{fr}}\, k^2}{\mathsf{s}_{\mathrm{B}}^2\mathfrak{a}^2}\, b_k\, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{B}}\, \mathfrak{a}) \;+\; \frac{2\,\mu_{\text{fr}}}{\mathsf{s}_{\mathrm{S}}\mathfrak{a}}\; c_k\, \omega\, \mathrm{i}k\, \mathrm{J}_k'(\omega\, \mathsf{s}_{\mathrm{S}}\, \mathfrak{a})\\
&+\omega^2\left(-\tfrac{2}{3}\mu_{\text{fr}} \;+ \mathrm{k}_{\text{fr}} \;+ M\alpha^2 + \alpha M \beta_{\mathrm{P}}\right)\, a_k \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) \;+\omega^2\left(-\tfrac{2}{3}\mu_{\text{fr}} + \mathrm{k}_{\text{fr}} + M\alpha^2 + \alpha M \beta_{\mathrm{B}}\right)\, b_k \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{B}}\, \mathfrak{a})\\
&\;= -\omega^2\, d_k\, \mathrm{H}_k^{(1)}(\omega\, \mathsf{s}_{\text{flu}}\, \mathfrak{a}) \;-\; \omega^2 \mathrm{i}^k\, J_k(\omega\, \mathsf{s}_{\text{flu}}\, \mathfrak{a})\; e^{-\mathrm{i}\, k\alpha_{\text{inc}}}\,,
\end{aligned}$$

$$
\begin{aligned}
&-\frac{2\,\mu_{\mathrm{fr}}\omega\,\mathrm{i}\,k}{\mathfrak{a}\,\mathsf{s}_{\mathrm{P}}}\,a_k\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,+\,\frac{2\,\mathrm{i}\,\mu_{\mathrm{fr}}\,k}{\mathfrak{a}^2\,\mathsf{s}_{\mathrm{P}}^2}\,a_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,-\,\frac{2\,\mu_{\mathrm{fr}}\omega\,\mathrm{i}\,k}{\mathfrak{a}\,\mathsf{s}_{\mathrm{B}}}\,b_k\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\\
&\quad+\,\frac{2\,\mathrm{i}\,\mu_{\mathrm{fr}}\,k}{\mathfrak{a}^2\,\mathsf{s}_{\mathrm{B}}^2}\,b_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,-\,\frac{\mu_{\mathrm{fr}}\,k^2}{\mathfrak{a}^2\,\mathsf{s}_{\mathrm{S}}^2}\,c_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,+\,\frac{\mu_{\mathrm{fr}}\,\omega}{\mathfrak{a}\,\mathsf{s}_{\mathrm{S}}}\,c_k\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\\
&\quad-\,\mu_{\mathrm{fr}}\,\frac{\omega}{\mathsf{s}_{\mathrm{S}}\mathfrak{a}}\,c_k\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,+\,\mu_{\mathrm{fr}}\,\frac{k}{\mathsf{s}_{\mathrm{S}}^2\mathfrak{a}^2}c_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\\
&\quad+\,\omega^2\,\mu_{\mathrm{fr}}\,c_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,-\,\mu_{\mathrm{fr}}\,\frac{k^2}{\mathsf{s}_{\mathrm{S}}^2\mathfrak{a}^2}\,c_k\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,=0\,.
\end{aligned}
$$

We can build a linear system satisfied by a_k, b_k, c_k, d_k in each mode k.

$$
\mathbb{A}_k^{\text{flu-poro}}\begin{pmatrix}a_k\\ b_k\\ c_k\\ d_k\end{pmatrix}=\begin{pmatrix}-\dfrac{\mathsf{s}_{\mathrm{flu}}}{\rho_{\mathrm{flu}}}\,\mathrm{i}^{k+1}\,J_k'(\omega\,\mathsf{s}_{\mathrm{flu}}\,\mathfrak{a})\;e^{-\mathrm{i}\,k\alpha_{\mathrm{inc}}}\\ -\mathrm{i}^k\,J_k(\omega\,\mathsf{s}_{\mathrm{flu}}\,\mathfrak{a})\;e^{-\mathrm{i}\,k\alpha_{\mathrm{inc}}}\\ -\,\omega^2\,\mathrm{i}^k\,J_k(\omega\,\mathsf{s}_{\mathrm{flu}}\,\mathfrak{a})\;e^{-\mathrm{i}\,k\alpha_{\mathrm{inc}}}\\ 0\end{pmatrix}
\tag{10.7}
$$

with

$$
\mathbb{A}_k^{\text{flu-poro}}=\begin{pmatrix}A_{11}&A_{12}&A_{13}&A_{14}\\ A_{21}&A_{22}&A_{23}&A_{24}\\ A_{31}&A_{32}&A_{33}&A_{34}\\ A_{41}&A_{42}&A_{43}&A_{44}\end{pmatrix}
\tag{10.8}
$$

$$
A_{11}=\frac{\mathfrak{s}\,\mathrm{i}}{\mathsf{s}_{\mathrm{P}}}\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,+\,\frac{\beta_{\mathrm{P}}\,\mathfrak{s}\,\mathrm{i}}{\mathsf{s}_{\mathrm{P}}}\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,,\qquad A_{12}=\frac{\mathfrak{s}\,\mathrm{i}}{\mathsf{s}_{\mathrm{B}}}\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,+\,\frac{\beta_{\mathrm{B}}\,\mathfrak{s}\,\mathrm{i}}{\mathsf{s}_{\mathrm{B}}}\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,,
$$

$$
A_{13}=\frac{\mathfrak{s}\,k}{\omega\,\mathsf{s}_{\mathrm{S}}^2\,\mathfrak{a}}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,-\,\frac{\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}}\,\mathfrak{s}\,k}{\det A\,\omega\,\mathfrak{a}}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,,\qquad A_{14}=\frac{\mathfrak{s}\,\mathsf{s}_{\mathrm{flu}}}{\rho_{\mathrm{flu}}}\,\mathrm{i}\,\mathrm{H}_k^{(1)'}(\omega\,\mathsf{s}_{\mathrm{flu}}\,\mathfrak{a})\,,
$$

$$
A_{21}=M\left(\beta_{\mathrm{P}}\;+\;\alpha\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,+\,\frac{\beta_{\mathrm{P}}\,\mathfrak{s}\,\mathrm{i}}{\kappa_\Gamma\,\mathsf{s}_{\mathrm{P}}}\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,,\quad A_{22}=M\left(\beta_{\mathrm{B}}\;+\;\alpha\right)\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,+\,\frac{\beta_{\mathrm{B}}\,\mathfrak{s}\,\mathrm{i}}{\kappa_\Gamma\,\mathsf{s}_{\mathrm{B}}}\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,,
$$

$$
A_{23}=-\frac{\mathfrak{s}\,k\,\rho_{\mathrm{f}}\,\mu_{\mathrm{fr}}}{\kappa_\Gamma\,\omega\,\det A\,\mathfrak{a}}\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,,\qquad A_{24}=\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{flu}}\,\mathfrak{a})\,,
$$

$$
\begin{aligned}
A_{31}=&-\frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{P}}\mathfrak{a}}\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,+\,\frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{P}}^2\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,+2\,\mu_{\mathrm{fr}}\,\omega^2\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,-\,\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{P}}^2\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\\
&+\,\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}}\,+\,\mathrm{k}_{\mathrm{fr}}\,+\,M\alpha^2\,+\,\alpha\,M\beta_{\mathrm{P}}\right)\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,,\\
A_{32}=&-\frac{2\,\mu_{\mathrm{fr}}\,\omega}{\mathsf{s}_{\mathrm{B}}\mathfrak{a}}\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,+\,\frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{B}}^2\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,+2\,\mu_{\mathrm{fr}}\,\omega^2\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,-\,\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{B}}^2\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\\
&+\,\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}}\,+\,\mathrm{k}_{\mathrm{fr}}\,+\,M\alpha^2\,+\,\alpha\,M\beta_{\mathrm{B}}\right)\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,,\\
A_{33}=&\,\frac{2\,\mu_{\mathrm{fr}}}{\mathsf{s}_{\mathrm{S}}\mathfrak{a}}\,\omega\,\mathrm{i}k\,\mathrm{J}_k^{'}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,,\qquad A_{34}=\omega^2\,\mathrm{H}_k^{(1)}(\omega\,\mathsf{s}_{\mathrm{flu}}\,\mathfrak{a})\,,
\end{aligned}
$$

and

$$
A_{41}=-\frac{2\,\mu_{\mathrm{fr}}\omega\,\mathrm{i}\,k}{\mathfrak{a}\,\mathsf{s}_{\mathrm{P}}}\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,+\,\frac{2\,\mathrm{i}\,\mu_{\mathrm{fr}}\,k}{\mathfrak{a}^2\,\mathsf{s}_{\mathrm{P}}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{P}}\,\mathfrak{a})\,,\quad A_{42}=-\frac{2\,\mu_{\mathrm{fr}}\omega\,\mathrm{i}\,k}{\mathfrak{a}\,\mathsf{s}_{\mathrm{B}}}\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,+\,\frac{2\,\mathrm{i}\,\mu_{\mathrm{fr}}\,k}{\mathfrak{a}^2\,\mathsf{s}_{\mathrm{B}}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{B}}\,\mathfrak{a})\,,
$$

$$
\begin{aligned}
A_{43}=&-\frac{\mu_{\mathrm{fr}}\,k^2}{\mathfrak{a}^2\,\mathsf{s}_{\mathrm{S}}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,+\,\frac{\mu_{\mathrm{fr}}\,\omega}{\mathfrak{a}\,\mathsf{s}_{\mathrm{S}}}\,\mathrm{J}_k'(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,-\,\mu_{\mathrm{fr}}\,\frac{\omega}{\mathsf{s}_{\mathrm{S}}\mathfrak{a}}\,\mathrm{J}_{k+1}(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,+\mu_{\mathrm{fr}}\,\frac{k}{\mathsf{s}_{\mathrm{S}}^2\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\\
&+\,\omega^2\,\mu_{\mathrm{fr}}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,-\,\mu_{\mathrm{fr}}\,\frac{k^2}{\mathsf{s}_{\mathrm{S}}^2\mathfrak{a}^2}\,\mathrm{J}_k(\omega\,\mathsf{s}_{\mathrm{S}}\,\mathfrak{a})\,,\\
A_{44}=&\,0\,.
\end{aligned}
$$

10.1.2 Open pores

If the hydraulic permeability $\kappa_\Gamma \to \infty$, the second equation becomes:

$$\begin{aligned} & d_k\, \mathrm{H}_k^{(1)}(\omega\, \mathsf{s}_{\mathrm{flu}}\, \mathfrak{a}) \;+\; \mathrm{i}^k\, J_k(\omega\, \mathsf{s}_{\mathrm{flu}}\, \mathfrak{a})\; e^{-\mathrm{i}\, k\alpha_{\mathrm{inc}}} \\ & -\big(-\, a_k\, M\, \big(\beta_{\mathrm{P}} \;+\; \alpha\big)\, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) -\; b_k\, M\, \big(\beta_{\mathrm{B}} \;+\; \alpha\big)\, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a})\big) \;=0\,. \end{aligned}$$

$$\mathbb{A}_k^{\text{flu-poro}} \begin{pmatrix} a_k \\ b_k \\ c_k \\ d_k \end{pmatrix} = \begin{pmatrix} -\dfrac{\mathsf{s}_{\mathrm{flu}}}{\rho_{\mathrm{flu}}}\, \mathrm{i}^{k+1}\, J_k'(\omega\, \mathsf{s}_{\mathrm{flu}}\, \mathfrak{a})\; e^{-\mathrm{i}\, k\alpha_{\mathrm{inc}}} \\ -\mathrm{i}^k\, J_k(\omega\, \mathsf{s}_{\mathrm{flu}}\, \mathfrak{a})\; e^{-\mathrm{i}\, k\alpha_{\mathrm{inc}}} \\ -\,\omega^2\, \mathrm{i}^k\, J_k(\omega\, \mathsf{s}_{\mathrm{flu}}\, \mathfrak{a})\; e^{-\mathrm{i}\, k\alpha_{\mathrm{inc}}} \\ 0 \end{pmatrix} \tag{10.9}$$

The second row is modified as:

$$A_{21} \,=\, M\, \big(\beta_{\mathrm{P}} \;+\; \alpha\big)\, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a})\,, \quad A_{22} \,=\, M\, \big(\beta_{\mathrm{B}} \;+\; \alpha\big)\, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{B}}\, \mathfrak{a})\,, \qquad A_{23} \,=\, 0,\,, \quad A_{24} \,=\, \mathrm{H}_k^{(1)}(\omega\, \mathsf{s}_{\mathrm{flu}}\, \mathfrak{a})\,.$$

10.1.3 Sealed pores

If the hydraulic permeability $\kappa_\Gamma = 0$, the second equation is:

$$a_k\, \frac{\beta_{\mathrm{P}}}{\mathsf{s}_{\mathrm{P}}}\, \omega\, \mathrm{J}_k'(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) \;+\; b_k\, \frac{\beta_{\mathrm{B}}}{\mathsf{s}_{\mathrm{B}}}\, \omega\, \mathrm{J}_k'(\omega\, \mathsf{s}_{\mathrm{B}}\, \mathfrak{a}) \;+\; c_k\, \frac{\rho_{\mathrm{f}}\, \mu_{\mathrm{fr}}}{\det A}\, \frac{\mathrm{i}k}{\mathfrak{a}} \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{S}}\, \mathfrak{a}) \;=\; 0\,.$$

$$\mathbb{A}_k^{\text{flu-poro}} \begin{pmatrix} a_k \\ b_k \\ c_k \\ d_k \end{pmatrix} = \begin{pmatrix} -\dfrac{\mathsf{s}_{\mathrm{flu}}}{\rho_{\mathrm{flu}}}\, \mathrm{i}^{k+1}\, J_k'(\omega\, \mathsf{s}_{\mathrm{flu}}\, \mathfrak{a})\; e^{-\mathrm{i}\, k\alpha_{\mathrm{inc}}} \\ 0 \\ -\,\omega^2\, \mathrm{i}^k\, J_k(\omega\, \mathsf{s}_{\mathrm{flu}}\, \mathfrak{a})\; e^{-\mathrm{i}\, k\alpha_{\mathrm{inc}}} \\ 0 \end{pmatrix} \tag{10.10}$$

The second row is modified as:

$$A_{21} \,=\, \frac{\beta_{\mathrm{P}}\, \omega}{\mathsf{s}_{\mathrm{P}}}\, \mathrm{J}_k'(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a})\,, \quad A_{22} \,=\, \frac{\beta_{\mathrm{B}}\, \omega}{\mathsf{s}_{\mathrm{B}}}\, \mathrm{J}_k'(\omega\, \mathsf{s}_{\mathrm{B}}\, \mathfrak{a})\,, \qquad A_{23} \,=\, \frac{\mathrm{i}\, k\, \rho_{\mathrm{f}}\, \mu_{\mathrm{fr}}}{\det A\, \mathfrak{a}} \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{S}}\, \mathfrak{a})\,, \quad A_{24} \,=\, 0\,.$$

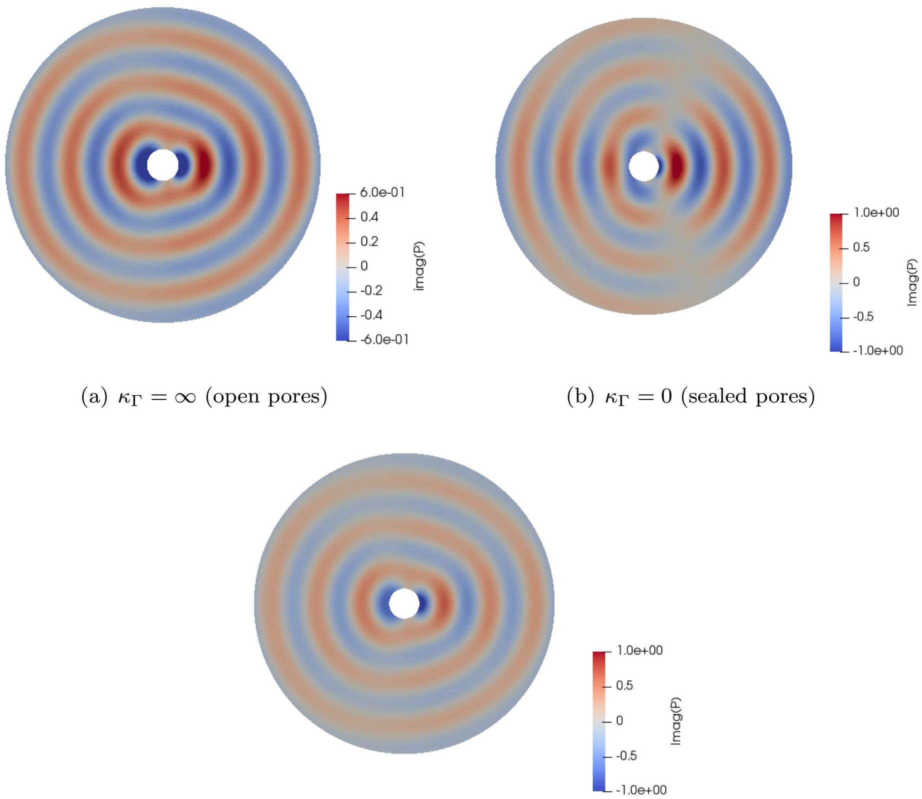
We define the eigenvalues as follows:

Definition 6. ω is fluid-porous Jones' mode corresponding to finite, open, sealed pores if $\det \mathbb{A}_k^{\text{flu-poro}}(\omega) = 0$, where $\mathbb{A}_k^{\text{flu-poro}}$ is the coefficients matrix defined in equations (10.8), (10.9) and (10.10) correspondingly.

10.2 Numerical tests

We study the trace of the pressure in the fluid using receivers on the radius $b = 8$m for $\omega = 500$ rad.s^{-1} and we sum the modes using $N_{sum} = 50$. Next, we are carrying out six experiments with two types of material configurations: sandstone with and without viscosity. For the three cases, we consider varying hydraulic permeability κ_Γ:

- Finite value of κ_Γ. Here we consider $\kappa_\Gamma = 1$.
- $\kappa_\Gamma = 0$, which corresponds to sealed pores.
- $\kappa_\Gamma = \infty$, which corresponds to open pores.



(a) $\kappa_\Gamma = \infty$ (open pores) (b) $\kappa_\Gamma = 0$ (sealed pores)

(c) $\kappa_\Gamma = 1$

Figure 18: Scattering of a fluid plane wave by a poroelastic inclusion. Reflected solution of the imaginary part of the pressure p for sandstone immersed in water with three values of the hydraulic permeability and no viscosity for $f = 500$ Hz.

For the first modes k, we will investigate the stability of the coefficient matrices. For all the tests, the cross section radius is $\mathfrak{a} = 1$m. The fluid parameters are given below:

$$\rho_{\text{flu}} = 10^3\, kg\,.\,m^{-3}\,, \quad \mathsf{s}_{\text{flu}} = 1500\, m\,.\,s^{-1}\,.$$

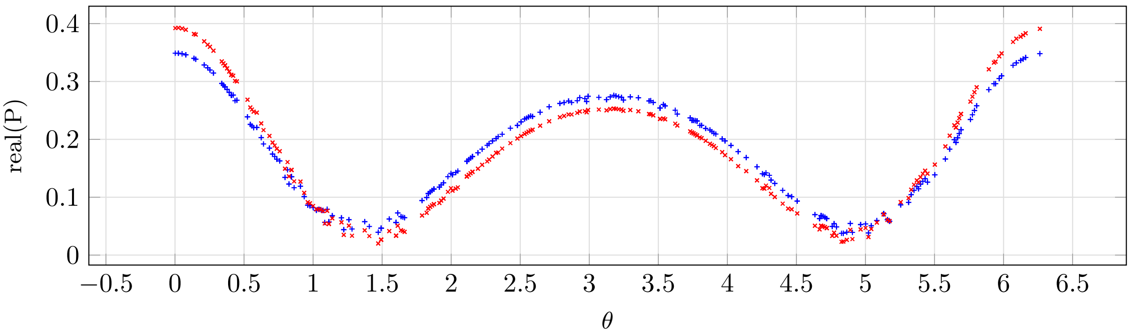
The results are respectively reported as follows:

- The trace of the pressure on a circle of radius $r = 8$m is presented in figure 19, for the three values of hydraulic permeability. For κ_Γ, we add the case of a fluid-elastic interaction problem. The parameters of the elastic material are described in section 11.
- Water/Sandstone ($\eta = 0$) in figures 21 and 22.
- Comparison of Water/Sandstone without viscosity and with viscosity $\eta \neq 0$ in figure 24.

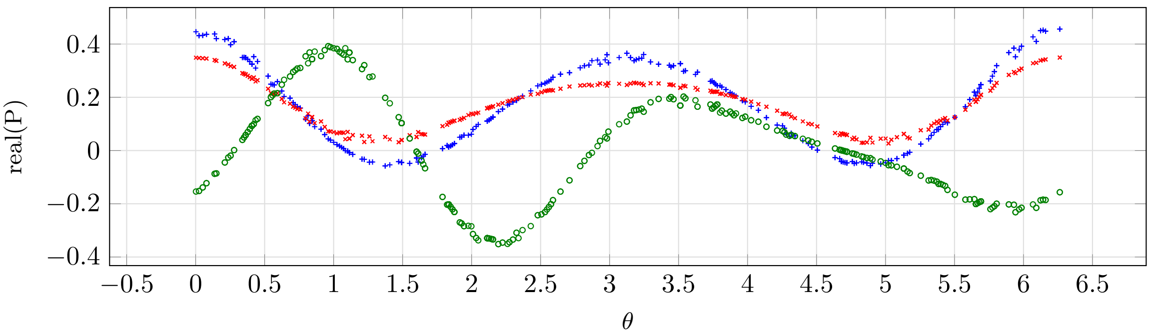
We have the following observations:

- In this case we observe for all the experiments with no viscosity the same behaviour. All the curves represent peaks, as in figures 21 and 22, but after zooming, some peaks have different behaviours. For most cases, the peaks are bounded below, hence there are no generalized eigenvalues. However, few of them are real eigenvalues, see figures 23. In the case with viscosity, the curves are smoother, we observe less peaks for the same range of frequency, and they are all bounded below after the zoom procedure.
- For $\kappa_\Gamma = 0$, we tested every peak of $\mathbb{A}_0^{\text{flu-poro}}$, for both media, because it is the case closest to elastic-fluid scattering. All the peaks are bounded below.

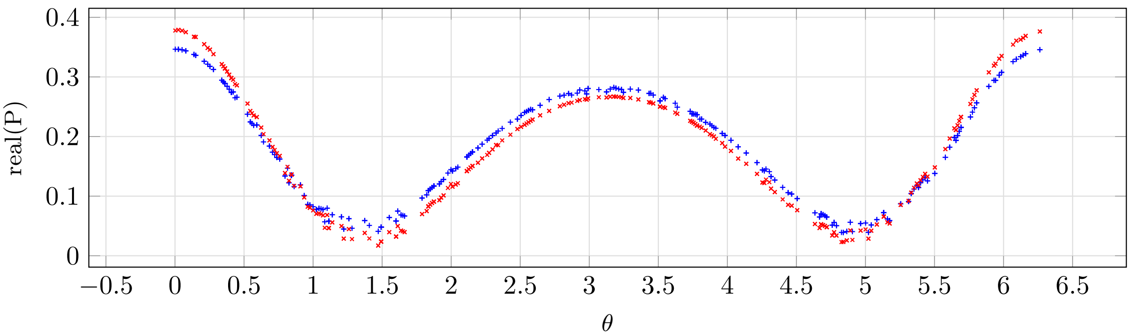
- The effect of the value of κ_Γ is limited. The cases $\kappa_\Gamma = 1$ and ∞ are similar, both for the value of the determinant and the value of the trace of the pressure.
- The comparison of the fluid-porous interaction with the fluid-elastic interaction highlights the fact that the behaviours are different, even though we took corresponding parameters between elastic and porous materials.



(a) $\kappa_\Gamma = 1$

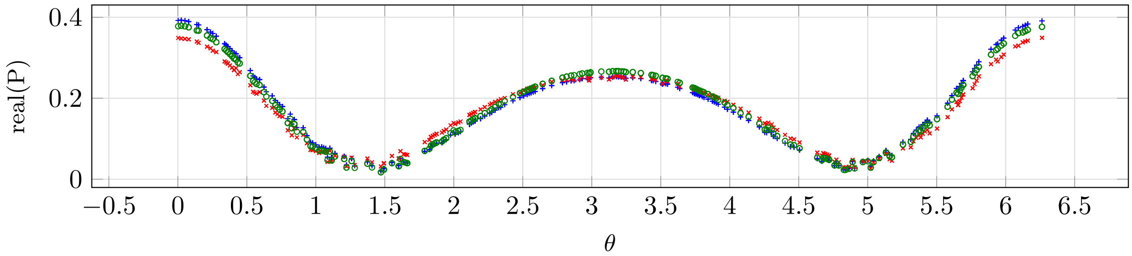


(b) $\kappa_\Gamma = 0$ (sealed pores)



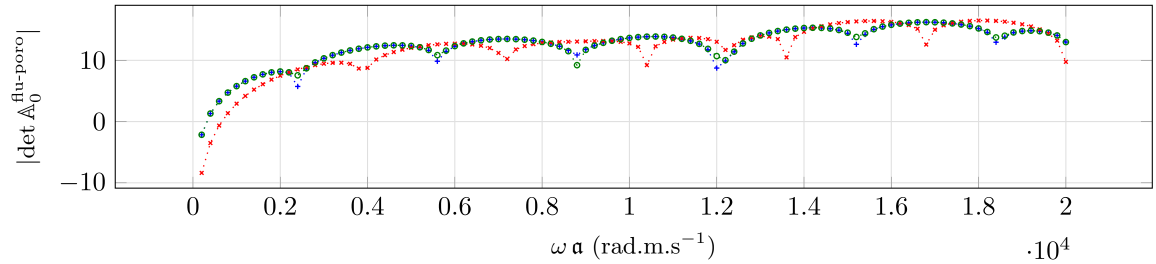
(c) $\kappa_\Gamma = \infty$ (open pores)

Figure 19: Experiment of a fluid-porous interaction. Comparison of the trace of the pressure for a sandstone solid immersed in water with and without viscosity and for several values of the hydraulic permeability. The test was computed for $f = 500$Hz. The case with no viscosity is represented in blue ×, while the case with viscosity $\eta = 10^{-3}$Pa.s is represented in red ∘. The case of an elastic solid is represented in green +.

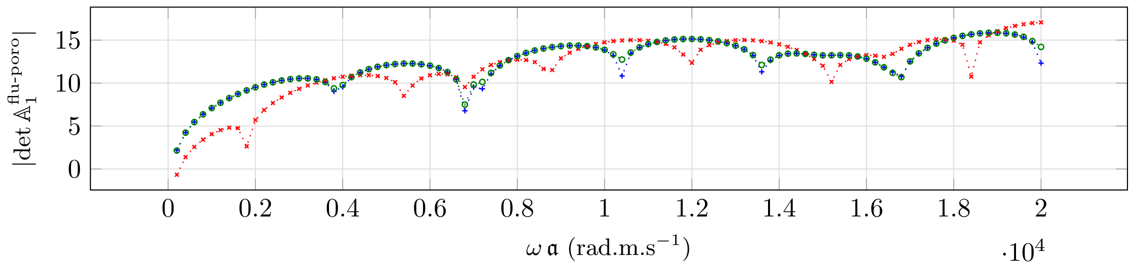


(a) $\eta = 10^{-3}$Pa.s

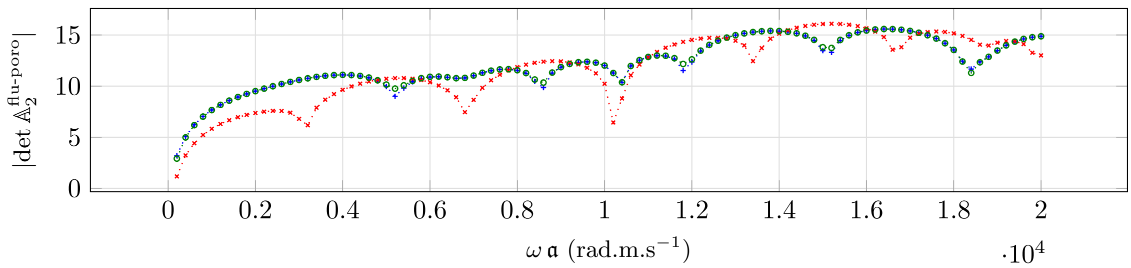
Figure 20: Experiment of a fluid-porous interaction. Comparison of the trace of the pressure for a sandstone solid immersed in water with viscosity $\eta = 10^{-3}$Pa.s for different values of the hydraulic permeability. The case $\kappa_\Gamma = 1$ is represented in blue, the case with sealed pores ($\kappa_\Gamma = 0$) is represented in red, and the case of open pores ($\kappa_\Gamma = \infty$) is represented in green.



(a) Mode $k = 0$

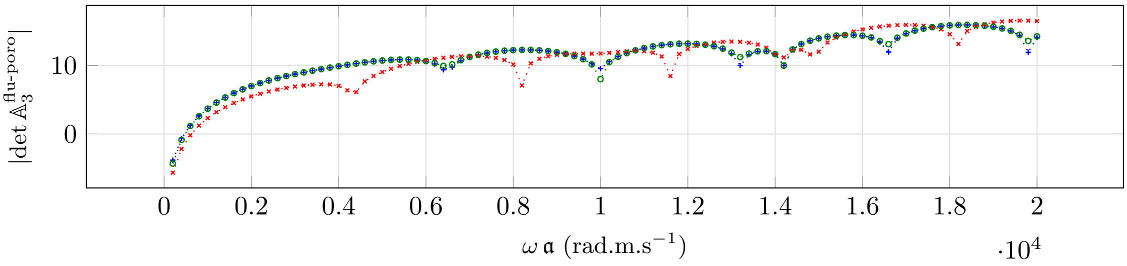


(b) Mode $k = 1$

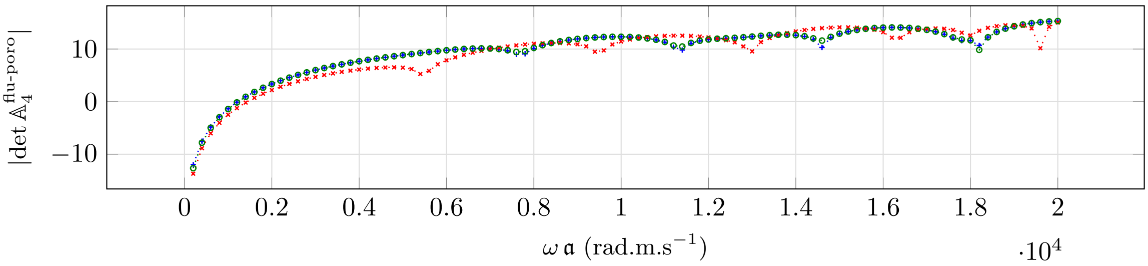


(c) Mode $k = 2$

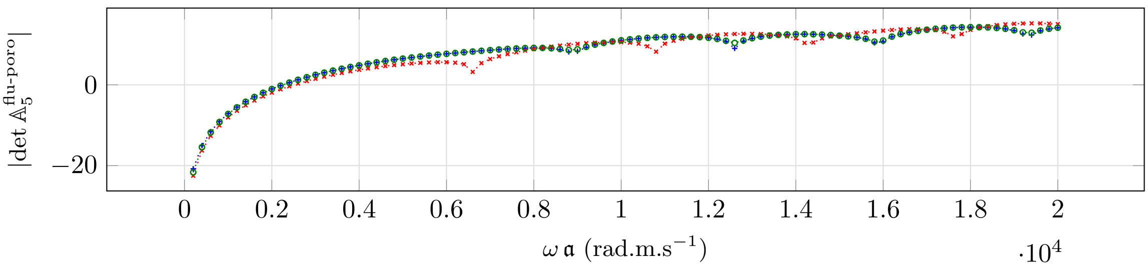
Figure 21: Experiment of a fluid-porous interaction. Determinant of the coefficients matrix $\mathbb{A}_k^{\text{water-sandstone}}$ (10.8) (log scale) for k in $0:2$ for sandstone with no viscosity $\eta = 0$. $\kappa_\Gamma = 1$ is represented in blue, $\kappa_\Gamma = 0$ in red and $\kappa_\Gamma = \infty$ in green.



(a) Mode $k = 3$

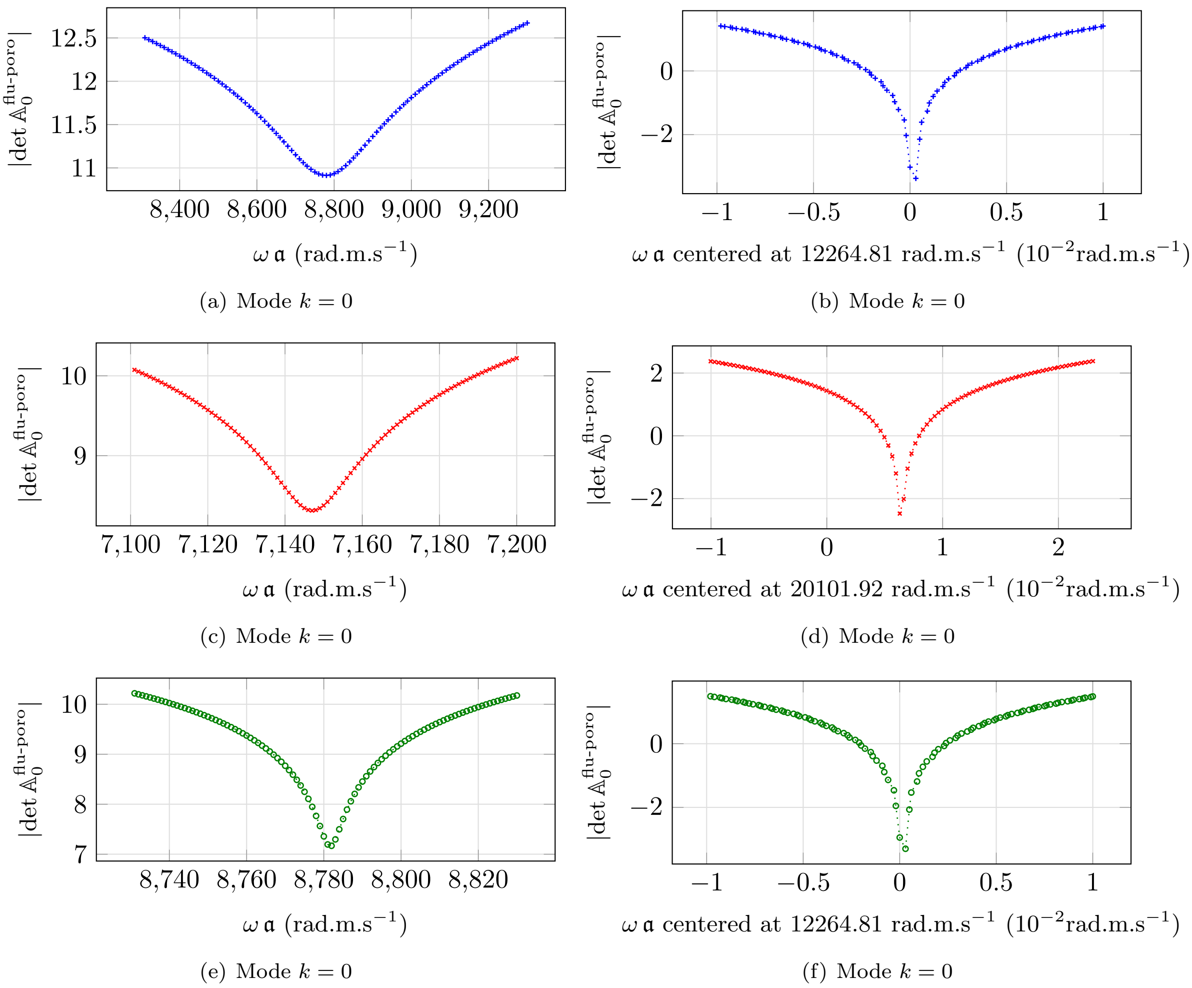


(b) Mode $k = 4$



(c) Mode $k = 5$

Figure 22: Experiment of a fluid-porous interaction. Determinant of the coefficients matrix $\mathbb{A}_k^{\text{water-sandstone}}$ (10.8) (log scale) for k in $3:5$, for sandstone with no viscosity $\eta = 0$. $\kappa_\Gamma = 1$ is represented in blue, $\kappa_\Gamma = 0$ in red and $\kappa_\Gamma = \infty$ in green.



(a) Mode $k = 0$

(b) Mode $k = 0$

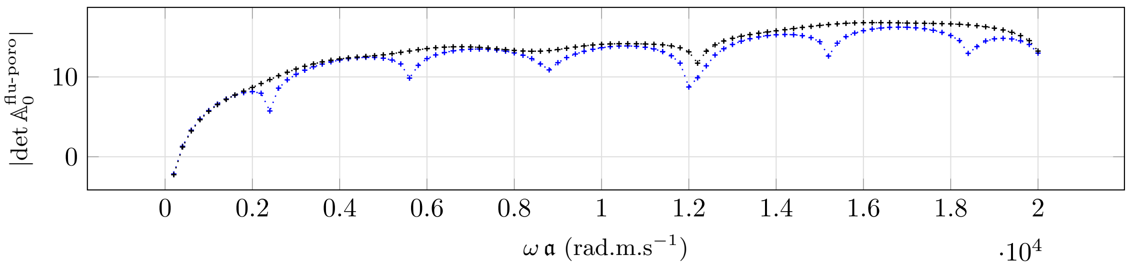
(c) Mode $k = 0$

(d) Mode $k = 0$

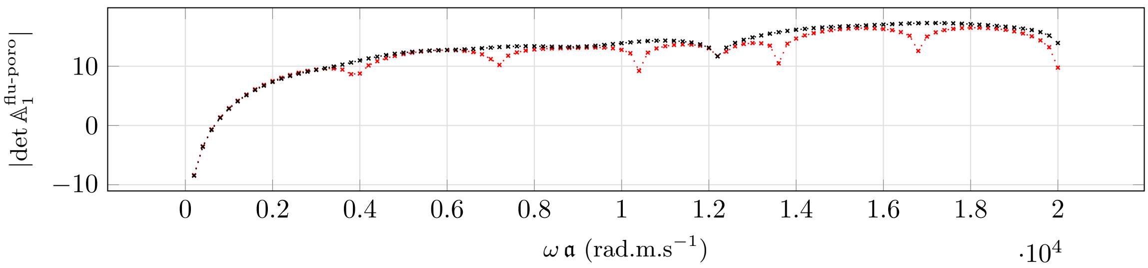
(e) Mode $k = 0$

(f) Mode $k = 0$

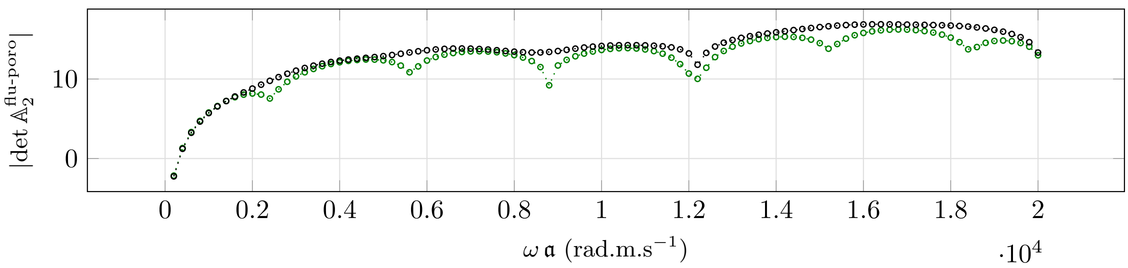
Figure 23: Experiment of a fluid-porous interaction. Determinant of the coefficients matrix $\mathbb{A}_k^{\text{water-sandstone}}$ (10.8) (log scale) zoomed on the frequency for $k = 0$ for sandstone with no viscosity $\eta = 0$. $\kappa_\Gamma = 1$ is represented in blue, $\kappa_\Gamma = 0$ in red and $\kappa_\Gamma = \infty$ in green.



(a) Mode $k = 0$, $\kappa_\Gamma = 1$



(b) Mode $k = 0$, $\kappa_\Gamma = 0$ (sealed pores)



(c) Mode $k = 0$, $\kappa_\Gamma = \infty$ (open pores)

Figure 24: Experiment of a fluid-porous interaction. Comparison of the determinant of the coefficients matrix $\mathbb{A}_k^{\text{water-sandstone}}$ (10.8) (log scale) for $k = 0$ for sandstone with and without viscosity for three values of hydraulic permeability. The cases with no viscosity are represented in color and the viscous cases are in black.

11 Comparison of the interaction problems

In this section, we compare the results obtained for interaction problems with two poroelastic domains (section 9), the interaction of a poroelastic domain in a fluid (section 10) and interaction of an elastic domain in an infinite porous medium. The fluid-solid interaction problem is described in [1]. We use the same matrix for the numerical tests, cf appendix D. The parameters used for the fluid-elastic test are given below:

$$\rho_{\text{flu}} = 10^3\, kg\,.\,m^{-3}\,,\;\; \mathsf{s}_{\text{flu}} = 1500\, m\,.\,s^{-1}\,,\;\; \rho_{\text{solid}} = 2.208\cdot 10^3\, kg\,.\,m^{-3}\,,\quad \lambda_{\text{solid}} = 32\, GPa\,,\;\text{ and }\; \mu_{\text{solid}} = 12\, GPa.$$

Note that the elastic parameters are taken to have a similar configuration between elastic and porous material. The solid density is taken equal to the average density of the porous medium. The value of the frame shear modulus of the elastic frame is the same as for sandstone, and the value of λ for the solid is the same as the undrained value for the sandstone.

Figure 25 represents the results for the following cases:

- Fluid/elastic interaction.
- Porous/porous interaction with the exterior medium composed of sandstone and the interior of shale (*cf.* table 1).
- Fluid/porous interaction where the fluid is water and the porous medium is sandstone. We have tested for three values of $\kappa_\Gamma = 0, 1$ and ∞. To simplify, we don't show the fluid-porous interaction with $\kappa_\Gamma = \infty$ because it is very close to the case with $\kappa_\Gamma = 1$.

The table 26 summarizes the results for all the tests described before.

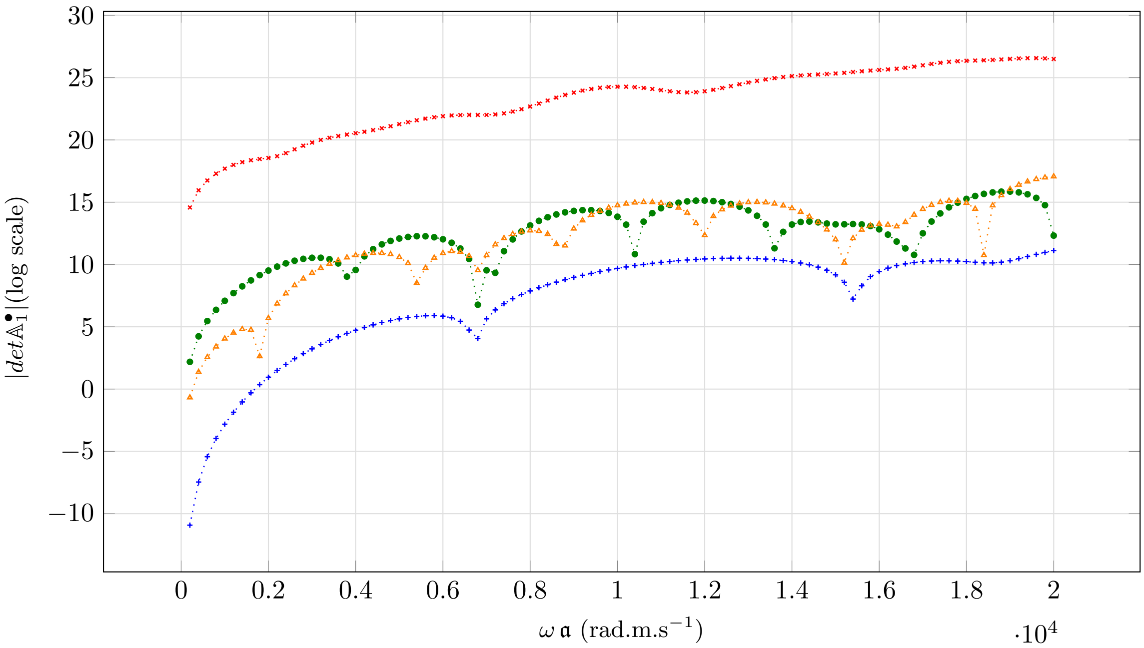


Figure 25: Comparison of the determinants of $\mathbb{A}_1$. We consider a sandstone medium with no viscosity. represents the fluid-elastic interaction, the porous-porous interaction for sandstone/shale, and [[Image Discarded Due To `'/tikz/external/mode=list and make'`]] the fluid-porous interaction for water/sandstone with respectively $\kappa_\Gamma = 0$ and 1.

	No viscosity $\eta = 0$	With viscosity $\eta \neq 0$
Porous-porous interaction	No generalized eigenvalues 14, 15	No generalized eigenvalues 14, 15
Fluid-porous interaction	No generalized eigenvalues 21	Presence of generalized eigenvalues 24
Fluid-elastic interaction	Presence of generalized eigenvalues 25	− − − − − − −

Figure 26: Summary of the results

12 Conclusion

We have computed analytical solutions in 2D for the following problems: bounded isotropic poroelastic problem, scattering of plane wave in poroelastic medium by penetrable/impenetrable obstacles and lastly fluid-solid interaction problem. As a first application of these formula, we give a description of a generic homogeneous solution for isotropic poroelastic equations, which includes outgoing solutions. As a second application, we carried out numerical investigations on the well-posedness of the above problems. The most interesting points that came out of this investigations are the followings:

- On bounded domain, the presence of eigenvalues does not come as a surprise when there is no viscosity. However, what is interesting is that, there are no eigenvalues with the current model of viscosity.

- In fluid-elastic interaction problems, the presence of what is equivalent with Jones' modes for poroelastic interior is found without viscosity. However, for the same range of frequency, for a viscous medium, there is no more eigenvalues.

However, these results are obtained for circular problems, they can be dependent on the considered geometry.

Current and future work We are currently preparing a report which makes use of this work to construct low order boundary absorbing conditions for isotropic poroelastic equations. Moreover, we will use the analytical solutions described in this work to perform analytical-numerical comparisons on an HDG method for poroelasticity. This study paves the way for future theoretical investigations of this question, such as the well-posedness of the outgoing solutions and theoretical confirmation of the absence of equivalent Jones' modes.

A Detailed calculation for expansion in Bessel functions

Denote by Z_k a Bessel function. To obtain the expressions (6.19) and (6.20) for $\mathbf{u}$ and $\mathbf{w}$, we use:

$$\mathbf{curl}\, f = \frac{1}{r}\,\partial_\theta f\,\mathbf{e}_r - \partial_r f\,\mathbf{e}_\theta\,, \quad \text{and} \quad \nabla f = \partial_r f\,\mathbf{e}_r + \frac{\partial_\theta}{r}\,\mathbf{e}_\theta\,,$$

to give

$$\begin{aligned}
\nabla\left(\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_\bullet\,|x|)e^{\mathrm{i}k\theta}\right) &= \tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_\bullet \mathrm{Z}_k'(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_\bullet\,|x|)e^{\mathrm{i}k\theta}\mathbf{e}_r + \frac{\mathrm{i}k}{|x|}\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_\bullet\,|x|)e^{\mathrm{i}k\theta}\mathbf{e}_\theta\,,\\
\mathbf{curl}\left(\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_\bullet\,|x|)e^{\mathrm{i}k\theta}\right) &= \frac{\mathrm{i}k}{|x|}\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_\bullet\,|x|)e^{\mathrm{i}k\theta}\;\mathbf{e}_r \;-\; \tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_\bullet\;\mathrm{Z}_k'(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_\bullet\,|x|)e^{\mathrm{i}k\theta}\mathbf{e}_\theta.
\end{aligned}$$

To express the components of $\boldsymbol{\tau}$, we will need the calculations of $\nabla^2 f$ and $\nabla(\mathrm{curl} f)$ for a scalar f:

$$\begin{aligned}
\nabla\left(\nabla f\right) &= \partial_r^2 f\,\mathbf{e}_r\otimes\mathbf{e}_r + \left(\frac{\partial_{r\theta}^2 f}{r} - \frac{\partial_\theta f}{r^2}\right)\mathbf{e}_r\otimes\mathbf{e}_\theta + \left(\frac{\partial_{r\theta}^2 f}{r} - \frac{\partial_\theta f}{r^2}\right)\mathbf{e}_\theta\otimes\mathbf{e}_r + \left(\frac{\partial_\theta^2 f}{r} + \frac{\partial_r f}{r}\right)\mathbf{e}_\theta\otimes\mathbf{e}_\theta\,,\\
\nabla\left(\mathbf{curl} f\right) &= \left(\frac{\partial_{r\theta}^2 f}{r} - \frac{\partial_\theta f}{r^2}\right)\mathbf{e}_r\otimes\mathbf{e}_r + \left(\frac{\partial_\theta^2 f}{r^2} + \frac{\partial_r f}{r}\right)\mathbf{e}_r\otimes\mathbf{e}_\theta - \partial_r^2 f\,\mathbf{e}_\theta\otimes\mathbf{e}_r + \left(-\frac{\partial_{r\theta}^2 f}{r} + \frac{\partial_\theta f}{r^2}\right)\mathbf{e}_\theta\otimes\mathbf{e}_\theta.
\end{aligned}$$

For $f = \mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_\bullet\,|x|)\,e^{\mathrm{i}\,k\,\theta}$,

$$\partial_r^2 f \;=\; (\omega\,\mathrm{s}_\bullet)^2\;\mathrm{Z}_k''(\tilde{\mathfrak{s}}\,\omega\,\mathrm{s}_\bullet|x|)\;e^{\mathrm{i}\,k\,\theta}\,,\quad \partial_{r\theta} f \;=\; \tilde{\mathfrak{s}}\,\omega\,\mathrm{s}_\bullet\,\mathrm{i}\,k\;\;\mathrm{Z}_k'(\tilde{\mathfrak{s}}\,\omega\,\mathrm{s}_\bullet|x|)\;e^{\mathrm{i}\,k\,\theta}\,,\quad \partial_\theta^2 f \;=\; -k^2\;\;\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathrm{s}_\bullet|x|)\;e^{\mathrm{i}\,k\,\theta}\,.$$

Next we replace Z'', by using the ODE:

$$z^2\frac{d^2}{dz^2}\mathrm{Z} \;+\; z\frac{d}{dz}\mathrm{Z} \;+\; (z^2 \;-\; k^2)\mathrm{Z} = 0 \quad \Rightarrow \frac{d^2}{dz^2}\mathrm{Z}_k \;=\; -\frac{1}{z}\mathrm{Z}_k' \;-\; (1-\frac{k^2}{z^2})\mathrm{Z}_k\,.$$

Remark 10. To completely eliminate the derivative, we can use the connection formula

$$\mathrm{Z}_k' = \mathrm{Z}_{k-1}(z) - \frac{k}{z}\mathrm{Z}_k(z) = -\mathrm{Z}_{k+1}(z) + \frac{k}{z}\mathrm{Z}_k(z)\,.$$

Hence, we have:

$$\begin{aligned}
\frac{d^2}{dz^2}\mathrm{Z}_k \;&=\; -\frac{1}{z}\left(\mathrm{Z}_{k-1}(z) - \frac{k}{z}\mathrm{Z}_k(z)\right) \;-\; (1-\frac{k^2}{z^2})\mathrm{Z}_k(z)\\
&=\; -\frac{1}{z}\left(-\mathrm{Z}_{k+1}(z) + \frac{k}{z}\mathrm{Z}_k(z)\right) \;-\; (1-\frac{k^2}{z^2})\mathrm{Z}_k(z)\,.
\end{aligned}$$

$\triangle$

Here, we detail the expression of the stress components $\boldsymbol{\tau}_{rr}$ and $\boldsymbol{\tau}_{r\theta}$:

$$\begin{aligned}
\omega^2\boldsymbol{\tau}_{rr} =\ & \mu_{\mathrm{fr}}\left(-\frac{2}{\mathsf{s}_{\mathrm{P}}^2}\,\partial_r^2\chi_{\mathrm{P}} \;-\; \frac{2}{\mathsf{s}_{\mathrm{B}}^2}\,\partial_r^2\chi_{\mathrm{B}} \;+\; 2\frac{\partial_{r\theta}\,\chi_{\mathrm{S}}}{\mathsf{s}_{\mathrm{S}}^2 r}\right) \;+\; \omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2\right)\,(\chi_{\mathrm{P}} \;+\; \chi_{\mathrm{B}}) + \;\omega^2\;\alpha\,M\,(\beta_{\mathrm{P}}\,\chi_{\mathrm{P}} \;+\; \beta_{\mathrm{B}}\,\chi_{\mathrm{B}})\\
=& \sum_{k\in\mathbb{Z}}\Bigg[\;\mu_{\mathrm{fr}}\left(-2\,a_k\,\omega^2\,\mathrm{Z}_k''(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{P}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} \;-\;2\,b_k\,\omega^2\,\mathrm{Z}_k''(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{B}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} + \frac{2}{\mathsf{s}_{\mathrm{S}} r}\,c_k\,\omega\,\tilde{\mathfrak{s}}\,\mathrm{i}k\,\mathrm{Z}_k'(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{S}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\right)\\
&+\;\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\,M\beta_{\mathrm{P}}\right)\;a_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{P}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\\
&+\;\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\,M\beta_{\mathrm{B}}\right)\,b_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{B}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\Bigg]\\
=& -\sum_{k\in\mathbb{Z}}\frac{2\,\mu_{\mathrm{fr}}\,\omega\,\tilde{\mathfrak{s}}}{\mathsf{s}_{\mathrm{P}}}\,a_k\,\mathrm{Z}_{k+1}(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{P}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}}\frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{P}}^2}\,a_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{P}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}} 2\,\mu_{\mathrm{fr}}\,a_k\,\omega^2\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{P}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\\
&-\sum_{k\in\mathbb{Z}}\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{P}}^2}\,a_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{P}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} - \sum_{k\in\mathbb{Z}}\frac{2\,\mu_{\mathrm{fr}}\,\omega\,\tilde{\mathfrak{s}}}{\mathsf{s}_{\mathrm{B}}}\,b_k\,\mathrm{Z}_{k+1}(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{B}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}}\frac{2\,\mu_{\mathrm{fr}}\,k}{\mathsf{s}_{\mathrm{B}}^2}\,b_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{B}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\\
&+\sum_{k\in\mathbb{Z}} 2\,\mu_{\mathrm{fr}}\,b_k\,\omega^2\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{B}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} - \sum_{k\in\mathbb{Z}}\frac{2\,\mu_{\mathrm{fr}}\,k^2}{\mathsf{s}_{\mathrm{B}}^2}\,b_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{B}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}}\frac{2\,\mu_{\mathrm{fr}}}{\mathsf{s}_{\mathrm{S}}^2 r}\,c_k\,\omega\,\mathsf{s}_{\mathrm{S}}\,\mathrm{i}k\,\mathrm{Z}_k'(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{S}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\\
&+\sum_{k\in\mathbb{Z}}\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\,M\beta_{\mathrm{P}}\right)\;a_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{P}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\\
&+\sum_{k\in\mathbb{Z}}\omega^2\,\left(-\tfrac{2}{3}\mu_{\mathrm{fr}} + \mathrm{k}_{\mathrm{fr}} + M\alpha^2 + \alpha\,M\beta_{\mathrm{B}}\right)\,b_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{B}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\,,
\end{aligned}$$

$$\begin{aligned}
\omega^2\,\boldsymbol{\tau}_{r\theta} \;=& \;\mu_{\mathrm{fr}}\left(-\frac{2}{\mathsf{s}_{\mathrm{P}}^2}\left(\frac{\partial_{\theta r}}{|x|}\chi_{\mathrm{P}} - \frac{\partial_\theta}{|x|^2}\chi_{\mathrm{P}}\right) \;-\; \frac{2}{\mathsf{s}_{\mathrm{B}}^2}\left(\frac{\partial_{\theta r}}{r}\chi_{\mathrm{B}} - \frac{\partial_\theta}{|x|^2}\chi_{\mathrm{B}}\right) \;+\; \frac{1}{\mathsf{s}_{\mathrm{S}}^2}\left(\frac{\partial_{\theta\theta}}{r^2}\,\chi_{\mathrm{S}} + \frac{\partial_r}{|x|}\chi_{\mathrm{S}} - \partial_{rr}\,\chi_{\mathrm{S}}\right)\right)\\
=& \sum_{k\in\mathbb{Z}}\mu_{\mathrm{fr}}\Bigg[-\frac{2}{\mathsf{s}_{\mathrm{P}}^2}\left(\frac{\omega\,\tilde{\mathfrak{s}}\,\mathsf{s}_{\mathrm{P}}\mathrm{i}\,k}{|x|}\,a_k\,\mathrm{Z}_k'(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{P}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} - \frac{\mathrm{i}\,k}{|x|^2}\,a_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{P}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\right)\\
&-\frac{2}{\mathsf{s}_{\mathrm{B}}^2}\left(\frac{\omega\,\tilde{\mathfrak{s}}\,\mathsf{s}_{\mathrm{B}}\mathrm{i}\,k}{|x|}\,b_k\,\mathrm{Z}_k'(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{B}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} - \frac{\mathrm{i}\,k}{|x|^2}\,b_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{B}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\right)\\
&+\frac{1}{\mathsf{s}_{\mathrm{S}}^2}\left(-\frac{k^2}{|x|^2}\,c_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{S}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} + \frac{\omega\,\tilde{\mathfrak{s}}\,\mathsf{s}_{\mathrm{S}}}{|x|}\,c_k\,\mathrm{Z}_k'(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{S}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} - \omega^2\,\mathsf{s}_{\mathrm{S}}^2\,c_k\,\mathrm{Z}_k''(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{S}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\right)\Bigg]\\
=& -\sum_{k\in\mathbb{Z}}\frac{2\,\mu_{\mathrm{fr}}\omega\,\tilde{\mathfrak{s}}\,\mathrm{i}\,k}{|x|\,\mathsf{s}_{\mathrm{P}}}\,a_k\,\mathrm{Z}_k'(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{P}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}}\frac{2\,\mathrm{i}\,\mu_{\mathrm{fr}}\,k}{|x|^2\,\mathsf{s}_{\mathrm{P}}^2}\,a_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{P}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\\
&-\sum_{k\in\mathbb{Z}}\frac{2\,\mu_{\mathrm{fr}}\omega\,\tilde{\mathfrak{s}}\,\mathrm{i}\,k}{|x|\,\mathsf{s}_{\mathrm{B}}}\,b_k\,\mathrm{Z}_k'(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{B}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}}\frac{2\,\mathrm{i}\,\mu_{\mathrm{fr}}\,k}{|x|^2\,\mathsf{s}_{\mathrm{B}}^2}\,b_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{B}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\\
&-\sum_{k\in\mathbb{Z}}\frac{\mu_{\mathrm{fr}}\,k^2}{|x|^2\,\mathsf{s}_{\mathrm{S}}^2}\,c_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{S}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}}\frac{\mu_{\mathrm{fr}}\,\omega\,\tilde{\mathfrak{s}}}{|x|\,\mathsf{s}_{\mathrm{S}}}\,c_k\,\mathrm{Z}_k'(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{S}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\\
&-\sum_{k\in\mathbb{Z}}\mu_{\mathrm{fr}}\,\frac{\omega\,\tilde{\mathfrak{s}}}{\mathsf{s}_{\mathrm{S}}}\,c_k\,\mathrm{Z}_{k+1}(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{S}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} + \sum_{k\in\mathbb{Z}}\mu_{\mathrm{fr}}\,\frac{k}{\mathsf{s}_{\mathrm{S}}^2}c_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{S}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\\
&+\sum_{k\in\mathbb{Z}}\mu_{\mathrm{fr}}\,\omega^2\,c_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{S}}\,|x|)\,e^{\mathrm{i}\,k\,\theta} - \sum_{k\in\mathbb{Z}}\mu_{\mathrm{fr}}\,\frac{k^2}{\mathsf{s}_{\mathrm{S}}^2}\,c_k\,\mathrm{Z}_k(\tilde{\mathfrak{s}}\,\omega\,\mathsf{s}_{\mathrm{S}}\,|x|)\,e^{\mathrm{i}\,k\,\theta}\,.
\end{aligned}$$

B Behavior in low frequency

B.1 Derivation of low-frequency equation

We obtain the equation under low-frequency assumption starting from Pride's general formula for dynamic permeability. Calculation is done in Convention 1. We recall [**?**, Eq (236)]

$$\frac{1}{k(\omega)} = \frac{1}{k_0}\Big(\sqrt{1-\mathrm{i}\frac{4}{m}\frac{\omega}{\omega_t}} - \mathrm{i}\frac{\omega}{\omega_t}\Big)\,.$$

Assume

$$\frac{\omega}{\omega_t}\frac{4}{m} \;\ll\; 1 \quad\Leftrightarrow\quad \frac{\omega}{\omega_t} \ll \frac{m}{4}\,.$$

Still in Pride's notation, the (dimensionless) number m and the transition frequency ω_t are defined as

$$m := \frac{\phi}{\alpha_\infty k_0}\Lambda^2 \quad , \quad \omega_t := \frac{\phi}{\alpha_\infty k_0}\frac{\eta}{\rho_\mathrm{f}} \,.$$

Recall that ω_t separates the low-frequency viscous-flow behavior from the high-frequency inertial flow. m is determined by experimental means, generally,

$$4 \leq m \leq 8 \,.$$

In [21], for numerical modeling, they use $m \equiv 6$. This means that

$$\omega \ll \frac{m}{4}\omega_t = \left(\frac{\phi}{\alpha_\infty k_0}\right)^2 \Lambda^2 \, \frac{\eta}{4\rho_\mathrm{f}} \,.$$

Under this assumption

$$\sqrt{1 - \mathrm{i}\frac{4}{m}\frac{\omega}{\omega_t}} = 1 - \mathrm{i}\frac{2}{m}\frac{\omega}{\omega_t} + \mathsf{O}\left(\left|\frac{4}{m}\frac{\omega}{\omega_t}\right|^2\right) ,$$

$$\Rightarrow \frac{1}{k(\omega)} = \frac{1}{k_0}\left[1 - \mathrm{i}\left(\frac{2}{m}+1\right)\frac{\omega}{\omega_t}\right] .$$

Putting this back in the equation, recall Darcy law in the equations of motion (3.12),

$$-\nabla \mathrm{p} = -\omega^2 \rho_\mathrm{f} \mathrm{u} - \omega^2 \tilde{\rho} \mathrm{w} \quad , \quad \tilde{\rho} = \frac{\mathrm{i}\eta}{\omega k(\omega)} \,. \tag{B.1}$$

Thus the term involving w can be written as

$$\begin{aligned} -\omega^2 \tilde{\rho}\mathrm{w} &= -\omega^2 \frac{\mathrm{i}\eta}{\omega k_0}\left[1 - \mathrm{i}\left(\frac{2}{m}+1\right)\frac{\omega}{\omega_t}\right]\mathrm{w} \\ &= -\omega \frac{\mathrm{i}\eta}{k_0}\left[1 - \mathrm{i}\left(\frac{2}{m}+1\right)\frac{\omega}{\omega_t}\right]\mathrm{w} \\ &= -\omega^2 \frac{\eta}{k_0\,\omega_t}\left(\frac{2}{m}+1\right)\mathrm{w} - \mathrm{i}\omega\frac{\eta}{k_0}\mathrm{w} \,. \end{aligned}$$

Consider the first coefficient, plugging in the definition of w_t and m according to Pride

$$\frac{\eta}{k_0\,\omega_t}\left(\frac{2}{m}+1\right) = \frac{\rho_\mathrm{f}}{\phi}\,\alpha_\infty \left(\frac{2}{m}+1\right) = \frac{\rho_\mathrm{f}}{\phi}\,\alpha_\infty \left(1 + 2k_0 \frac{\alpha_\infty}{\phi}\frac{1}{\Lambda^2}\right) .$$

Thus the equation (B.1) in low frequency is

$$-\nabla \mathrm{p} = -\omega^2 \rho_\mathrm{f} \mathrm{u} - \omega^2 \rho_\mathrm{w}\,\mathrm{w} - \mathrm{i}\omega\frac{\eta}{k_0}\mathrm{w} \quad ; \quad \rho_\mathrm{w} := \frac{\rho_\mathrm{f}}{\phi}\,\alpha_\infty\left(\frac{2}{m}+1\right) .$$

For consistencies, the 'generic tortuosity' in low-frequency should be

$$\mathsf{t} := \alpha_\infty \left(\frac{2}{m}+1\right) , \tag{B.2}$$

a multiple of Pride's tortuosity α_∞. This agrees with [39, p.14][4]

Equations of motion in low-frequency With

$$\rho_\mathrm{w} := \frac{\rho_\mathrm{f}}{\phi}\mathsf{t} \quad , \quad \mathsf{t} := \alpha_\infty\left(\frac{2}{m}+1\right) . \tag{B.3}$$

$$\begin{array}{c} \textbf{Low-freq} \\ \textit{(convention 1)} \end{array} \quad \begin{aligned} \nabla\cdot\boldsymbol{\tau} + \mathbf{f}_\mathrm{u} &= -\omega^2\,\rho_a\,\mathrm{u} \;-\; \omega^2\,\rho_\mathrm{f}\,\mathrm{w} \,, \\ -\nabla\mathrm{p} + \mathbf{f}_\mathrm{w} &= -\omega^2 \rho_\mathrm{f}\,\mathrm{u} - \omega^2\rho_\mathrm{w}\,\mathrm{w} - \mathrm{i}\omega\frac{\eta}{k_0}\mathrm{w} \quad . \end{aligned} \tag{B.4}$$

Due to Darcy's law, and put in the form using $\tilde{\rho}$, the equation of motion can be written as

$$\begin{array}{c} \textbf{Low-freq} \\ \textit{(convention 1)} \end{array} \quad \begin{aligned} -\nabla\mathrm{p} + \mathbf{f}_\mathrm{w} &= -\omega^2 \rho_\mathrm{f}\,\mathrm{u} - \omega^2 \tilde{\rho}_\mathrm{LF}(\omega)\,\mathrm{w} \,, \\ \text{with} \quad \tilde{\rho}_\mathrm{LF}(\omega) &= \rho_\mathrm{w} + \mathrm{i}\frac{\eta}{\omega\,k_0} = \mathrm{i}\frac{\eta}{\omega}\left(-\mathrm{i}\omega\,\frac{\rho_\mathrm{w}}{\eta} + k_0\right) . \end{aligned} \tag{B.5}$$

[4]In the notation [39, p.14] $\frac{2}{m} \to \phi$ and $\rho_\mathrm{w} \to \rho_e$.

For convention 2, we have:

$$
\begin{aligned}
&\textbf{Low-freq} && \nabla\cdot\boldsymbol{\tau}+\mathbf{f}_{\mathrm{u}} &&= -\omega^2\,\rho_a\,\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{f}}\,\mathrm{w}\,,\\
&\textit{(convention 2)} && -\nabla\mathrm{p}+\mathbf{f}_{\mathrm{w}} &&= -\omega^2\rho_{\mathrm{f}}\,\mathrm{u}-\omega^2\rho_{\mathrm{w}}\,\mathrm{w}+\mathrm{i}\omega\frac{\eta}{k_0}\mathrm{w}\\
& && &&= -\omega^2\rho_{\mathrm{f}}\,\mathrm{u}-\omega^2\tilde{\tilde{\rho}}_{\mathrm{LF}}(\omega)\,\mathrm{w}\,,\\
& && \text{with}\quad \tilde{\tilde{\rho}}_{\mathrm{LF}}(\omega) &&= \rho_{\mathrm{w}}-\mathrm{i}\frac{\eta}{\omega\,k_0} \;=\; -\mathrm{i}\frac{\eta}{\omega}\left(\mathrm{i}\omega\,\frac{\rho_{\mathrm{w}}}{\eta}+k_0\right)\quad .
\end{aligned}
\tag{B.6}
$$

We unify these two forms by using the low-frequency limit defined in (3.20) of the dynamic density ρ_{dyn} (3.15)

$$
\rho_{\mathrm{dyn}}^{\mathrm{LF}} \;:=\; \rho_{\mathrm{w}} \;+\; \mathfrak{s}\,\mathrm{i}\frac{\eta}{\omega\,k_0} \;=\; \mathrm{i}\frac{\eta}{\omega}\left(-\mathrm{i}\omega\,\frac{\rho_{\mathrm{w}}}{\eta}\;+\;\mathfrak{s}\,k_0\right).
$$

B.2 Some discussion on the limit at vanishing viscosity for low-frequency regime

Here we start from the low-frequency equation of motion (3.21), and consider its formal limit at vanishing viscosity. To obtain 'formally' the form of the equations at vanishing viscosity i.e. $\eta\to 0$, we *do not* put the constraint $\omega<\omega_t$. Note that ω_t is a constant multiple of η. Thus technically, when $\eta\to 0$, $\omega_t\to 0$ (assuming other physical parameters such as α_∞, k_0, ϕ, Λ appearing in the definition of constant m and ω_t in (3.8)) are independent of the shear viscosity η. Recall the equations of motion in low-frequency (B.5) in convention 1

$$
-\nabla\mathrm{p}+\mathbf{f}_{\mathrm{w}} \;=\; -\omega^2\rho_{\mathrm{f}}\,\mathrm{u}-\omega^2\,\rho_{\mathrm{dyn}}^{\mathrm{LF}}(\omega)\,\mathrm{w}\,,
$$

with

$$
\rho_{\mathrm{dyn}}^{\mathrm{LF}}(\omega) \;=\; \rho_{\mathrm{w}}+\mathrm{i}\frac{\eta}{\omega\,k_0}=\mathrm{i}\frac{\eta}{\omega}\left(-\mathrm{i}\omega\,\frac{\rho_{\mathrm{w}}}{\eta}+k_0\right),
$$

and parameters given by (B.3)

$$
\rho_{\mathrm{w}} \;:=\; \frac{\rho_{\mathrm{f}}}{\phi}\mathsf{t}\quad,\quad \mathsf{t}:=\alpha_\infty\left(\frac{2}{m}+1\right)\quad,\quad m:=\frac{\phi}{\alpha_\infty k_0}\Lambda^2\,.
$$

In taking limits, we assume that the quantities $\phi,\alpha_\infty,k_0,\Gamma,m$ are independent of η, we have

$$
\lim_{\substack{\eta\to 0,\\ \text{fixed }\omega>0}}\tilde{\tilde{\rho}}_{\mathrm{LF}}(\omega) \;=\; \lim_{\substack{\eta\to 0,\\ \text{fixed }\omega>0}}\rho_{\mathrm{dyn}}^{\mathrm{LF}}(\omega) \;=\; \rho_{\mathrm{w}} \;=\; \frac{\rho_{\mathrm{f}}}{\phi}\alpha_\infty\left(\frac{2}{m}+1\right).
\tag{B.7}
$$

For both conventions, low-frequency with no viscosity, the equation of motions are given by

$$
\begin{aligned}
&\textbf{Low-frequency zero-viscosity} && \nabla\cdot\boldsymbol{\tau}+\mathbf{f}_{\mathrm{u}} &&= -\omega^2\,\rho_a\,\mathrm{u} \;-\; \omega^2\,\rho_{\mathrm{f}}\,\mathrm{w}\,,\\
&(\textit{both conventions}) && -\nabla\mathrm{p}+\mathbf{f}_{\mathrm{w}} &&= -\omega^2\rho_{\mathrm{f}}\,\mathrm{u}-\omega^2\rho_{\mathrm{w}}\,\mathrm{w}\,.
\end{aligned}
\tag{B.8}
$$

C Potential method vs. Helmholtz decomposition

For elasticity, the unknown displacement is written as functions of the scalars ψ_H and $\tilde{\psi}_H$,

$$
\mathrm{u} \;=\; \nabla\psi_H \;+\; \mathbf{curl}\,\tilde{\psi}_H,
$$

cf. [28] which follows [38] and [30], or [23] and [16] which follows [29]. For poroelasticity, this decomposition is applied to both u and w,

$$
\mathrm{u} \;=\; \nabla\psi_H \;+\; \mathbf{curl}\,\tilde{\psi}_H \quad,\quad \mathrm{w} \;=\; \nabla\phi_H \;+\; \mathbf{curl}\,\tilde{\phi}_H,
$$

cf. [15] which follows [6] or [7]. Although our unknowns, see (6.7) below, are also scalar functions and called potentials, there are some slight differences between the two approaches. We do not impose the Helmholtz decomposition on the original unknowns u and w, but exploit the very specific form of the poroelastic equation, which in fact forces a generic solution to have this decomposition. In terms of the Helmholtz potentials, our unknowns (6.7) are $\Delta\psi_H$, $\Delta\phi_H$, $\mathbf{curl}\,\tilde{\psi}_H$, and $\mathbf{curl}\,\tilde{\phi}_H$, see derivation in appendix C. The system of PDE, see (6.8a)–(6.8b), satisfied by the potentials are obtained in a less cumbersome manner than substituting the Helmholtz decomposition into the equation. For elasticity, see the comparison with (C.3), and for poroelasticity with (C.6). We also note that [6] and [7] work with nonzero sources, and our aim for this report is zero sources, since we work with scattering of plane waves and not point sources. The second slight difference is how the system is treated, we use diagonalization (in Step 2b) and still work with a system of PDE of second order (for the pressure potentials), while in [6], that for pressure (Helmholtz) potentials is a fourth order equation,

$$
(\Delta+\mathsf{k}_{\mathrm{P}}^2)(\Delta+\mathsf{k}_{\mathrm{S}}^2)\psi=0,
$$

cf. Eq (6) in [6]. See also [36, Eq (42)], which solves the equation with no source but however uses the Helmhotz decompsition.

C.1 Elasticity

We consider the homogeneous isotropic elastic equation with unknown u denoting the solid displacement

$$\omega^2\,\rho\,u + (\,\lambda\,+2\,\mu)\,\nabla\,\nabla\,\cdot u + \mu\,\mathbf{curl}\,\mathrm{curl}\,u = 0\,. \tag{C.1}$$

Helmholtz decomposition approach: In Helmholtz decomposition approach, analytic expressions are obtained by first using Helmholtz decomposition, i.e. writing u as

$$u \;=\; \nabla\psi \;+\; \mathbf{curl}\,\tilde{\psi}\,, \tag{C.2}$$

e.g. cf. [28] which follows [38] and [30], or [23] and [16] which follows[5] [29]. We next substitute the decomposition (C.2) into the elastic equation (C.1)

$$\omega^2\rho\,(\nabla\psi \;+\; \mathbf{curl}\,\tilde{\psi}) \;+\; (\lambda \;+\; 2\mu)\nabla\,\nabla\cdot(\nabla\psi \;+\; \mathbf{curl}\,\tilde{\psi}) \;+\; \mu\,\mathbf{curl}\,\mathrm{curl}\,(\nabla\psi \;+\; \mathbf{curl}\,\tilde{\psi}) \;=\; 0\,.$$

Using $\nabla\cdot\,\mathbf{curl} = 0$ and $\mathrm{curl}\nabla = 0$, this simplifies and is equivalent to

$$\begin{aligned}
&\omega^2\rho(\nabla\psi \;+\; \mathbf{curl}\,\tilde{\psi}) \;+\; (\lambda+2\mu)\nabla\,\nabla\cdot\nabla\psi \;+\; \mu\,\mathbf{curl}\,\mathrm{curl}\,\mathbf{curl}\,\tilde{\psi} \;=\; 0\,,\\
\Leftrightarrow\;& \big(\omega^2\rho\nabla\psi \;+\; (\lambda+2\mu)\nabla\,\nabla\cdot\nabla\psi\big) \;+\; \big(\omega^2\,\rho\,\mathbf{curl}\,\tilde{\psi} \;+\; \mu\,\mathbf{curl}\,\mathrm{curl}\,\mathbf{curl}\,\tilde{\psi}\big) = 0\,.
\end{aligned}$$

We next require each of the two quantities in the above expression to be zero. Note that in this step, this breaks the equivalence of the equation:

$$\omega^2\rho\nabla\psi \;+\; (\lambda+2\mu)\nabla\,\nabla\cdot\nabla\psi = 0 \quad\Leftrightarrow\quad \omega^2\rho\nabla\psi \;+\; (\lambda+2\mu)\nabla\,\Delta\psi = 0\,,$$

and

$$\omega^2\rho\,\mathbf{curl}\,\tilde{\psi} \;+\; \mu\,\mathbf{curl}\,\mathrm{curl}\,\mathbf{curl}\,\tilde{\psi} = 0 \quad\Leftrightarrow\quad \omega^2\,\rho\,\mathbf{curl}\,\tilde{\psi} \;-\; \mu\,\mathbf{curl}\,\Delta\tilde{\psi} \;=\; 0\,.$$

Assuming ρ, λ and μ constant, this is equivalent to

$$\nabla\Big(\omega^2\rho\psi \;+\; (\lambda+2\mu)\,\Delta\psi\Big) \;=\; 0 \quad\text{and}\quad \mathbf{curl}\,(\omega^2\,\rho\,\tilde{\psi} \;-\; \mu\,\Delta\tilde{\psi}) = 0\,.$$

We require further that

$$\omega^2\rho\,\psi \;+\; (\lambda+2\mu)\,\Delta\psi \;=\; 0 \quad\text{and}\quad \omega^2\rho\,\tilde{\psi} \;-\; \mu\,\Delta\tilde{\psi} \;=\; 0\,. \tag{C.3}$$

Note that without further assumption this step also breaks the equivalence with previous equation. From (C.3), we deduce that ψ and $\tilde{\psi}$ have to be of the form

$$\psi \;=\; \sum_{k\in\mathbb{Z}} a_n\,\mathrm{Z}_n(\mathsf{k}_{\mathrm{P}}\,r) \quad ,\tilde{\psi} \;=\; \sum_{k\in\mathbb{Z}} \tilde{a}_n\,\mathrm{Z}_n(\mathsf{k}_{\mathrm{S}}\,r)\,,$$

for $\mathrm{Z}_k = \mathrm{J}_k$ the Bessel function if u is defined as a solution on a disc, or $\mathrm{Z}_k = \mathrm{H}^{(1)}_k$ the first Hankel function if u is an outgoing solution. We obtain the 'necessary' form for the displacement u

$$u \;=\; \nabla\left(\sum_{k\in\mathbb{Z}} a_n\,\mathrm{Z}_n(\mathsf{k}_{\mathrm{P}}r)\right) \;+\; \mathbf{curl}\left(\sum_{k\in\mathbb{Z}} b_n\,\mathrm{Z}_n(\mathsf{k}_{\mathrm{S}}r)\right)\,.$$

However, this does not guarantee that u will solve the elastic equation (C.1).

---

[5]In obtaining their analytic solution for the fluid-solid interaction problem, [23] and [16] start right-away with an Ansatz for the displacement vector of the elastic core, given by [29]

$$u = \nabla\psi - \mathbf{e}_z \times (\nabla_{(x,y,z)}\,\psi)\,.$$

Since $\mathbf{e}_z \times \nabla_{(x,y,z)}\psi = -\mathbf{curl}\,\psi$, this Ansatz is the same as the one given by the Helmholtz decomposition. From observing the expansion of the acoustic planewave in Bessel J_n and with only $\cos(n\theta)$ in the radial part, and carrying out separation of variables in cylindrical coordinates for the elastic equation, [29] (and hence [23] and [16]) further imposes the following form of $\psi_\bullet$, *cf.* [29, Eqn 4.1–4.2 p 280]

$$\psi_{\mathrm{P}} = \sum_{k=0}^{\infty} c_n\,\mathrm{J}_n(\mathsf{k}_{\mathrm{P}}r)\,\cos(k\theta) \quad , \quad \psi_{\mathrm{S}} = \sum_{k=0}^{\infty} d_n\,\mathrm{J}_n(\mathsf{k}_{\mathrm{P}}r)\,\sin(k\theta)\,.$$

This poses as another difference from our approach, since we only carry out separation of variables in cylindrical coordinates for the scalar Helmholtz equation, which is much simpler.

Potential theory approach In the potential theory approach, the elastic equation (C.1) can be *equivalently* written as

$$u \;=\; -\frac{\lambda + 2\mu}{\omega^2\,\rho}\nabla\varphi \;+\; \mu\,\mathbf{curl}\,\tilde{\varphi}\,, \tag{C.4}$$

where

$$\varphi := \nabla\cdot u \quad \text{and} \quad \tilde{\varphi} := \mathrm{curl} u\,.$$

Now taking $\nabla\cdot$ (C.4) and curl (C.4), we obtain two equations for φ and $\tilde{\varphi}$

$$\varphi \;=\; -\frac{\lambda + 2\mu}{\omega^2\,\rho}\Delta\varphi \quad \text{and} \quad \tilde{\varphi} \;=\; \mu\,\Delta\,\tilde{\varphi}\,.$$

Note that this also breaks the equivalence of the equation[6].

As φ and $\tilde{\varphi}$ are solutions of Helmholtz equation, they can be expanded in terms of Bessel functions:

$$\varphi \;=\; \sum_{k\in\mathbb{Z}} b_n\,\mathrm{Z}_n(\mathsf{k}_\mathrm{P}\,r)\,, \quad \tilde{\varphi} \;=\; \sum_{k\in\mathbb{Z}} \tilde{b}_n\,\mathrm{Z}_n(\mathsf{k}_\mathrm{S}\,r)\,.$$

We substitute this into the necessary form of a solution u given in equation (C.4),

$$u \;=\; -\frac{\lambda + 2\mu}{\omega^2\,\rho}\sum_{k\in\mathbb{Z}} b_n\,\nabla\mathrm{Z}_n(\mathsf{k}_\mathrm{P}\,r) \;+\; \mu\,\sum_{k\in\mathbb{Z}} \tilde{b}_n\,\mathbf{curl}\,\mathrm{Z}_n(\mathsf{k}_\mathrm{S}\,r)\,. \tag{C.5}$$

C.2 Helmholtz decomposition for Poroelasticity

In Section 6, we used potential theory to solve the homogeneous poroelastic equation. Here, we use Helmholtz decomposition. We will see that there are very slight differences between the two approaches. However the potential exposition is less cumbersome. Using the Helmholtz decomposition of u and w

$$u \;=\; \nabla\psi \;+\; \mathbf{curl}\,\tilde{\psi} \qquad w \;=\; \nabla\phi \;+\; \mathbf{curl}\,\tilde{\phi},$$

and substitute these into the equation (3.27), we obtain:

$$\begin{aligned} -\omega^2\rho_a\mathrm{u} \;-\; \rho_\mathrm{f}\omega^2\mathrm{w} \;-\; H\,\nabla\nabla\cdot\mathrm{u} \;+\; \mu_\mathrm{fr}\,\mathbf{curl}\,\mathrm{curl}\,\mathrm{u} \;-\; \alpha\,M\,\nabla\nabla\cdot\mathrm{w} = \mathbf{f}_\mathrm{u}\,,\\ -\omega^2\rho_\mathrm{f}\mathrm{u} \;-\; \omega^2\,\rho_\mathrm{dyn}(\omega)\,\mathrm{w} - M\,\nabla\nabla\cdot\mathrm{w} \;-\; M\alpha\nabla\nabla\cdot\mathrm{u} = \mathbf{f}_\mathrm{w} \;+\; \nabla M\,\mathbf{f}_\mathrm{p}\,. \end{aligned}$$

The variables are now the Helmholtz potentials, ψ , $\tilde{\psi}$, ϕ , and $\tilde{\phi}$. Using the following identities in 2D for function f and vector $\mathbf{v}$,

$$\begin{aligned} \nabla\cdot\mathbf{curl} \;=\; 0 \quad &, \quad \mathrm{curl}\,\nabla \;=\; 0\,,\\ \mathrm{curl}\,\mathbf{curl}\,f \;=\; -\Delta f \quad &, \quad \mathbf{curl}\,\mathrm{curl}\,\mathbf{v} \;=\; \nabla\,\nabla\cdot\mathbf{v} \;-\; \Delta\mathbf{v}\,, \end{aligned}$$

we have

$$\begin{aligned} \nabla\nabla\cdot u = \nabla\nabla\cdot\nabla\psi = \nabla\Delta\psi \quad &, \quad \nabla\nabla\cdot w = \nabla\Delta\phi\,,\\ \mathbf{curl}\,\mathrm{curl}\,u = \mathbf{curl}\,\mathrm{curl}\,\mathbf{curl}\,\tilde{\psi} &= -\mathbf{curl}\Delta\tilde{\psi}\,. \end{aligned}$$

The poroelastic equation becomes

$$\begin{aligned} -\omega^2\rho_a(\nabla\psi \;+\; \mathbf{curl}\,\tilde{\psi}) \;-\; \rho_\mathrm{f}\omega^2(\nabla\phi \;+\; \mathbf{curl}\,\tilde{\phi}) \;-\; H\,\nabla\Delta\psi \;-\; \mu_\mathrm{fr}\,\mathbf{curl}\Delta\tilde{\psi} \;-\; \alpha\,M\,\nabla\Delta\phi = \mathbf{f}_\mathrm{u}\,,\\ -\omega^2\rho_\mathrm{f}(\nabla\psi \;+\; \mathbf{curl}\,\tilde{\psi}) \;-\; \omega^2\,\rho_\mathrm{dyn}(\omega)\,(\nabla\phi \;+\; \mathbf{curl}\,\tilde{\phi}) - M\,\nabla\Delta\phi \;-\; M\alpha\nabla\Delta\psi = \mathbf{f}_\mathrm{w} \;+\; \nabla M\,\mathbf{f}_\mathrm{p}\,. \end{aligned}$$

After rearrangement, we obtain

$$\begin{aligned} \nabla(-\omega^2\rho_a\psi \;-\; \rho_\mathrm{f}\omega^2\,\phi \;-\; H\,\Delta\psi \;-\; \alpha\,M\,\Delta\phi) + \mathbf{curl}(-\omega^2\rho_a\,\tilde{\psi} \;-\; \rho_\mathrm{f}\omega^2\,\tilde{\phi} \;-\; \mu_\mathrm{fr}\,\Delta\tilde{\psi}) = \mathbf{f}_\mathrm{u}\,,\\ \nabla(-\omega^2\rho_\mathrm{f}\psi \;-\; \omega^2\,\rho_\mathrm{dyn}\,\phi - M\,\Delta\phi \;-\; M\alpha\Delta\psi) + \mathbf{curl}(-\omega^2\rho_\mathrm{f}\,\tilde{\psi} \;-\; \omega^2\,\rho_\mathrm{dyn}\,\tilde{\phi}) = \mathbf{f}_\mathrm{w} \;+\; \nabla M\,\mathbf{f}_\mathrm{p}\,. \end{aligned} \tag{C.6}$$

In the case of no sources, we require that the potentials are such that the expressions in the four parentheses are zero, and obtain the system of PDE (6.8a)–(6.8b) obtain in proof of Prop 6,

$$\begin{aligned} -\omega^2\rho_a\psi \;-\; \rho_\mathrm{f}\omega^2\,\phi \;-\; H\,\Delta\psi \;-\; \alpha\,M\,\Delta\phi = 0\,,\\ -\omega^2\rho_\mathrm{f}\psi \;-\; \omega^2\,\rho_\mathrm{dyn}\,\phi - M\,\Delta\phi \;-\; M\alpha\Delta\psi = 0\,,\\ -\omega^2\rho_a\,\tilde{\psi} \;-\; \rho_\mathrm{f}\omega^2\,\tilde{\phi} \;-\; \mu_\mathrm{fr}\,\Delta\tilde{\psi} = 0\,,\\ -\omega^2\rho_\mathrm{f}\,\tilde{\psi} \;-\; \omega^2\,\rho_\mathrm{dyn}\,\tilde{\phi} = 0\,. \end{aligned}$$

[6] Having zero divergence and zero curl does not guarantee that a vector is zero.

In the potential method in Section 6, we work with the variables $\operatorname{curl} u$, $\operatorname{curl} w$, $\nabla \cdot u$ and $\nabla \cdot w$. They are related to the Helmholtz potential by,

$$\begin{aligned} \operatorname{curl} u \,=\, \operatorname{curl} \mathbf{curl}\, \tilde{\psi} \,=\, -\Delta\, \tilde{\psi} \quad , \qquad & \operatorname{curl} w = \operatorname{curl} \mathbf{curl}\, \tilde{\phi} = -\Delta\, \tilde{\phi}, \\ \nabla \cdot u = \Delta\, \psi \quad , \qquad & \nabla \cdot w = \Delta \phi \,. \end{aligned} \tag{C.7}$$

D Coefficients matrix for an elastic solid inclusion immersed in a fluid medium

We consider an elastic cylindric inclusion immersed in a fluid. The section of the inclusion is a disc of radius $\mathfrak{a}$. The expression of the pressure is

$$\mathrm{p} = \sum_{k \in \mathbb{Z}} a_k \, \mathrm{H}_k^{(1)}(\omega\, \mathsf{s}_{\text{fluid}}\, |\boldsymbol{x}|)\; e^{\mathrm{i}\, k\, \theta}\,,$$

and the velocity value is

$$\mathbf{u} = \nabla_{(x,y)} \sum_{k \in \mathbb{Z}} b_k \, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P}}\, |\boldsymbol{x}|)\; e^{\mathrm{i}\, k\, \theta} - e_z \times \nabla_{(x,y,z)} \sum_{k \in \mathbb{Z}} c_k \, \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{S}}\, |\boldsymbol{x}|)\; e^{\mathrm{i}\, k\, \theta}\,,$$

with

$$\mathsf{s}_{\mathrm{P}} = \sqrt{\frac{\rho_{\text{solid}}}{\lambda + 2\mu}} \qquad \text{and} \qquad \mathsf{s}_{\mathrm{S}} = \sqrt{\frac{\rho_{\text{solid}}}{\lambda}}\,.$$

We denote $k_{\text{fluid}} = \omega\, \mathsf{s}_{\text{fluid}}$, $k_P = \omega\, \mathsf{s}_{\mathrm{P}}$, and $k_S = \omega\, \mathsf{s}_{\mathrm{S}}$. The coefficient matrices is: [1]

$$\mathbb{A}^k = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}\,,$$

with

$$A_{11} = k_{\text{fluid}} \mathrm{H}_k^{(1)'}(\omega\, \mathsf{s}_{\text{fluid}}\, \mathfrak{a})\,, \qquad A_{12} = -\omega^2\, \rho_{\text{fluid}} k_P \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a})\,, \qquad A_{13} = -\omega^2\, \rho_{\text{fluid}} k_S \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{S}}\, \mathfrak{a})\,,$$

$$A_{21} = \mathrm{H}_k^{(1)}(\omega\, \mathsf{s}_{\text{fluid}}\, \mathfrak{a})\,, \qquad A_{22} = \frac{2\,\mu}{\mathfrak{a}^2}\left((n^2 \,+ n \,-\, \frac{1}{2} k_P^2\, \mathfrak{a}^2) \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) -\, k_P \mathrm{J}_{k-1}(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) \right)\,,$$

$$A_{23} = \frac{2\,\mu}{\mathfrak{a}^2} k \left(-(n-1) \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{S}}\, \mathfrak{a}) + k_S \mathfrak{a} \mathrm{J}_{k-1}(\omega\, \mathsf{s}_{\mathrm{S}}\, \mathfrak{a}) \right)\,, \qquad A_{31} = 0\,,$$

$$A_{32} = \frac{2\,\mu}{\mathfrak{a}^2} k \left(-(n-1) \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) + k_P \mathfrak{a} \mathrm{J}_{k-1}(\omega\, \mathsf{s}_{\mathrm{P}}\, \mathfrak{a}) \right)\,, \qquad A_{33} = \frac{2\,\mu}{\mathfrak{a}^2}\left((n^2 \,+ n \,-\, \frac{1}{2} k_S^2\, \mathfrak{a}^2) \mathrm{J}_k(\omega\, \mathsf{s}_{\mathrm{S}}\, \mathfrak{a}) -\, k_S \mathrm{J}_{k-1}(\omega\, \mathsf{s}_{\mathrm{S}}\, \mathfrak{a}) \right)\,.$$

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